Financial Theory IV: Term Structure Models

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Overview

- Models with affine stochastic discount factors are:
  - General
  - Highly tractable
  - Straightforward to estimate
- Often used to price bonds (term structure).
  - But can also be used to price general assets like equity.
- Overview of today's lecture:
  1. Term structure models of bonds
  2. Models of the equity term structure.
  3. General affine SDF models of asset prices.
  4. Case Study: "How the Wealth Was Won"

Affine Term Structure Models of Bonds

Bond Prices

- Focus on zero-coupon bonds (or strips)
- $P_{n,t}$ is the price of an n-period zero coupon bond:

$$P_{n,t} = E_t \left(\prod_{i=1}^{n} M_{t+i} \right)$$

- Prices of other securities with *deterministic* cash flows follow mechanically. For instance, cash flows $\{CF_{t+i}\}$ ($i = 1, \ldots, n$):

$$\sum_{i=1}^{n} P_{i,t} CF_{t+i}$$

- Log bond prices are denoted by $p_{n,t} = \log P_{n,t}$

Yields

- The n-period, continuously compounded yield is defined as:

$$P_{n,t} = \exp\left(-y_{n,t} n\right),$$

  or:

$$y_{n,t} = -\frac{1}{n} p_{n,t}$$

- Note: inverse relationship between prices and yields
- Current yield curve:

  `http://www.bloomberg.com/markets/rates-bonds/government-bonds/us/`

Returns and Forward Rates

- The holding-period return on an n-period bond is defined as:

$$\begin{aligned} r_{n,t+1} &= p_{n-1,t+1} - p_{n,t} \\ &= n y_{n,t} - (n-1) y_{n-1,t+1} \end{aligned}$$

- The forward rate allows you to borrow and lend at some future point in time and a pre-specified interest rate, $f_t^{n \to n+1}$

$$\begin{aligned} f_t^{n \to n+1} &= p_{n,t} - p_{n+1,t} \\ &= (n+1) y_{n+1,t} - n y_{n,t} \\ &= y_{n+1,t} + n \left(y_{n+1,t} - y_{n,t} \right) \end{aligned}$$

$\implies$ Forward rates higher than yields if the term structure is increasing

- Note: $f_t^{0 \to 1} = y_{1,t}$

Forward rates

- We can always write:

$$\begin{aligned} p_{n,t} &= p_{n,t} - p_{n-1,t} + p_{n-1,t} - \ldots + p_{1,t} \\ &= -\left(f_t^{n-1\to n} + f_t^{n-2\to n-1} + \cdots + y_{1,t}\right) \end{aligned}$$

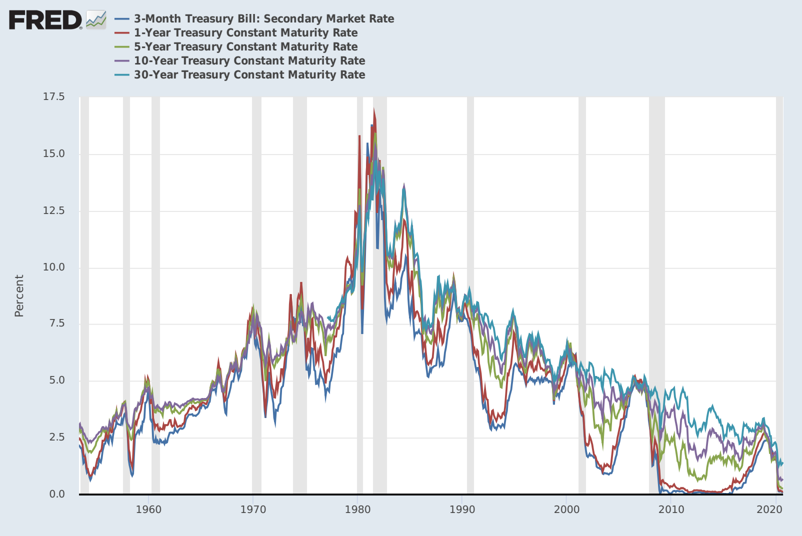
- This implies:

$$p_{n,t} = -\sum_{i=0}^{n-1} f_t^{i\to i+1}$$

or:

$$P_{n,t} = \exp\left(-\sum_{i=0}^{n-1} f_t^{i\to i+1}\right)$$

Yields on US Treasuries



Expectations Hypothesis

- Theory for understanding the *shape* of the yield curve
- Three (mathematically) equivalent statements
  1. $y_{n,t} = \frac{1}{n} E_t \left(y_{1,t} + ... + y_{1,t+n-1} \right) \; [+RP_1]$
  2. $f_t^{(n \to n+1)} = E_t \left(y_{1,t+n} \right) \; [+RP_2]$
  3. $E_t \left(p_{n-1,t+1} - p_{n,t} \right) = y_{1,t} \; [+RP_3]$
- Pure Expectations Hypothesis: $RP_1 = RP_2 = RP_3 = 0$
- Generalized Expectations Hypothesis: there is a risk premium, but it is constant over time
- PEH = risk neutrality (up to Jensen terms)

Discrete Time Term Structure Models

- We will consider four workhorse models

1. Model based on the expectations hypothesis
2. Vasicek model
3. Cox-Ingersoll-Ross model
4. $k-$factor essentially affine model in discrete time

Model 1: Expectations Hypothesis

- Pure expectations hypothesis:

$$y_{n,t} = \frac{1}{n} E_t \left\{ \sum_{j=0}^{n-1} y_{1,t+j} \right\}$$

- Stochastic process for short rate (state variable):

$$y_{1,t+1} = \delta + \rho \left(y_{1,t} - \delta \right) + \varepsilon_{t+1}$$

- Combine:

$$\begin{aligned} y_{n,t} &= \delta + \frac{1}{n} \sum_{j=0}^{n-1} E_t \left(y_{1,t+j} - \delta \right) = \delta + \frac{1}{n} \sum_{j=0}^{n-1} \rho^j \left(y_{1,t} - \delta \right) \\ &= \delta + \frac{1}{n} \frac{1 - \rho^n}{1 - \rho} \left(y_{1,t} - \delta \right) \end{aligned}$$

- What structure would deliver this outcome?

Model 1: Expectations Hypothesis

- Pure expectations hypothesis:

$$y_{n,t} = \frac{1}{n} E_t \left\{ \sum_{j=0}^{n-1} y_{1,t+j} \right\}$$

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- What structure would deliver this outcome?

Expectations Hypothesis: Properties

- One-factor model that can generate upward and downward sloping term structures
- On average, there is no slope: $E\left(y_{n,t}\right) = \delta, \ \forall n$
- All bond yields are perfectly correlated
- Interest rates can become negative
- Yields and bond returns are homoskedastic

General SDF Approach

1. State variables x_t evolve according to: $x_{t+1} = h(x_t, \varepsilon_{t+1})$
2. Define short rate as a function of the state variables: $y_{1,t} = g(x_t)$
3. The SDF takes the form

$$M_{t+1} = \frac{\exp\left(-y_{1,t} - \lambda(x_t)'\varepsilon_{t+1}\right)}{E_t \exp\left(-\lambda(x_t)'\varepsilon_{t+1}\right)}$$

4. Solve bond prices recursively using

$$P_{n+1,t} = E_t\left[M_{t+1}P_{n,t+1}\right]$$

with initial condition $P_{0,t} = 1$.

Model 2: Vasicek Model

- Similar to EH, but introduces a constant bond risk premium
- Single factor:

$$x_{t+1} = \rho x_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}\left(0, \sigma^2\right)$$

- Short rate is affine in this single factor:

$$y_{1,t} = \delta + x_t$$

- Can interpret x as level factor and ε as shocks to the level of the term structure.
- Instead of imposing the expectations hypothesis, we specify the SDF:

$$M_{t+1} = \exp\left(-y_{1,t} - \frac{1}{2}\lambda^2\sigma^2 - \lambda\varepsilon_{t+1}\right)$$

Vasicek: Bond Prices

- Recursively construct bond prices of the form

$$P_{n,t} = \exp(A_n + B_n x_t).$$

- Start with $n = 1$:

$$P_{1,t} = \exp\left(-y_{1,t}\right) = \exp\left(A_1 + B_1 x_t\right),$$

  so that

$$\begin{aligned} A_1 &= -\delta \\ B_1 &= -1. \end{aligned}$$

Vasicek: Bond Prices

- For general $n+1$:

$$
\begin{aligned}
P_{n+1,t} &= E_t\left(M_{t+1}P_{n,t+1}\right) \\
&= E_t\left(\exp\left(-\delta - x_t - \frac{1}{2}\lambda^2\sigma^2 - \lambda\varepsilon_{t+1} + A_n + B_n x_{t+1}\right)\right) \\
&= E_t\left(\exp\left(-\delta - x_t - \frac{1}{2}\lambda^2\sigma^2 - \lambda\varepsilon_{t+1} + A_n + B_n(\rho x_t + \varepsilon_{t+1})\right)\right) \\
&= \exp\left(-\delta - x_t + A_n + B_n\rho x_t + \frac{1}{2}B_n^2\sigma^2 - \lambda B_n\sigma^2\right) \\
&= \exp\left(A_{n+1} + B_{n+1}x_t\right),
\end{aligned}
$$

with:

$$
\begin{aligned}
A_n &= -\delta + A_{n-1} + \frac{1}{2}B_{n-1}^2\sigma^2 - \lambda B_{n-1}\sigma^2, \\
B_n &= -1 + B_{n-1}\rho
\end{aligned}
$$

Vasicek: Bond Prices and Yields

- These difference equations can be solved recursively, and in closed-form in this simple case

$$B_n = -\frac{1-\rho^n}{1-\rho}$$

$$A_n = -n\delta + \sum_{j=1}^{n-1} \left(\frac{1}{2} B_{n-1}^2 \sigma^2 - \lambda B_{n-1} \sigma^2 \right)$$

- Note: can solve partial sums using lag operator plus indicator:

$$B_n = \mathbf{1}_{\{n>1\}} \left(\rho B_{n-1} - 1 \right)$$

$$(1 - \rho L \mathbf{1}_{\{n>1\}}) B_n = -\mathbf{1}_{\{n>1\}}$$

$$B_n = -\sum_{j=0}^{\infty} \rho^j L^j \mathbf{1}_{\{n>1\}}^{j+1} = -\sum_{j=0}^{n-1} \rho^j = -\frac{1-\rho^n}{1-\rho}$$

Vasicek: Properties

- One-factor model that can generate upward and downward sloping term structures
- Holding period return:

$$
\begin{aligned}
r_{n+1,t+1} &= p_{n,t+1} - p_{n+1,t} \\
&= \delta + x_t - \frac{1}{2} B_n^2 \sigma^2 + \lambda B_n \sigma^2 + B_n \varepsilon_{t+1}
\end{aligned}
$$

  sum of short rate, Jensen term, risk premium, and innovation.

- Risk premium is constant, positive if $\lambda < 0$

Properties

- Bond yields all perfectly correlated, given by

$$y_{n,t} = -\frac{A_n}{n} - \frac{B_n}{n} x_t$$

- Interest rates can become negative
- Yields and bond returns are homoskedastic
- Auto-correlation:
  - Because all yields and yield spreads are linear combinations of the short rate, they inherit their autocorrelations from $y_{1,t}$
  - Long yields have higher autocorrelations in the data, yield spreads have lower autocorrelations

Model 3: Cox-Ingersoll-Ross

- Single state variable follows square root process:

$$x_{t+1} = \delta + \rho \left(x_t - \delta \right) + \sqrt{x_t} \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0, \sigma^2)$$

- The short rate is equal to this single factor:

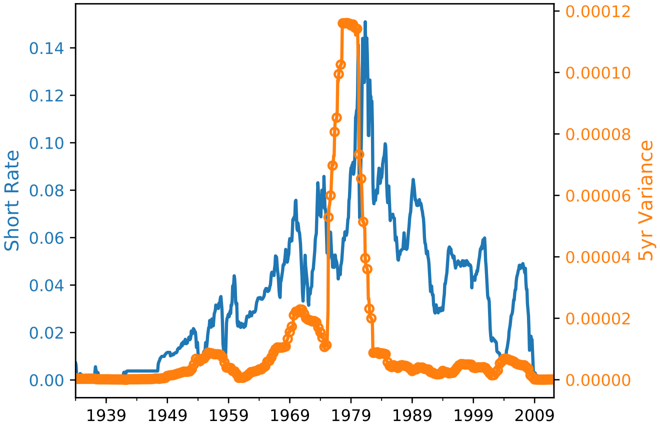
$$y_{1,t} = x_t$$

- We specify the SDF:

$$M_{t+1} = \exp\left(-y_{1,t} - \frac{1}{2}\lambda^2 x_t \sigma^2 - \lambda \sqrt{x_t} \varepsilon_{t+1} \right)$$

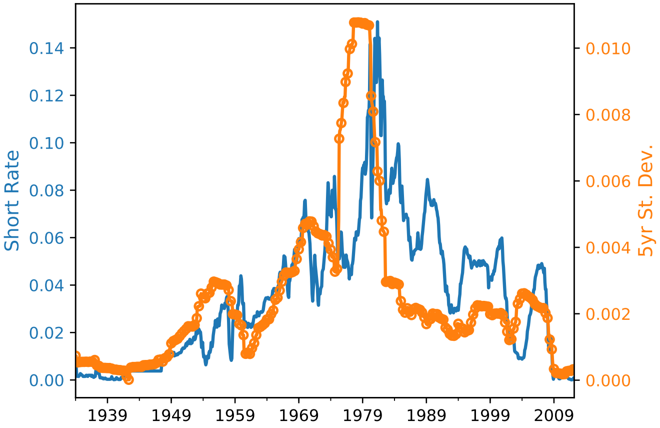
Square Root Process: Evidence

- 3mo rate vs. 5y-ahead realized variance:



Square Root Process: Evidence

- 3mo rate vs. 5y-ahead realized standard deviation:



Cox-Ingersoll-Ross: Bond Prices

- Conjecture that the log price of an $n+1$-period bond can be expressed as:

$$P_{n,t} = \exp(-ny_{n,t}) = \exp(A_n + B_n x_t)$$

- Note that $A_1 = 0$ and $B_1 = -1$ from the short rate equation

- Verify from the Euler equation

Cox-Ingersoll-Ross: Bond Prices

- Conjecture that the log price of an $n+1$-period bond can be expressed as:

$$P_{n,t} = \exp(-n y_{n,t}) = \exp(A_n + B_n x_t)$$

- Note that $A_1 = 0$ and $B_1 = -1$ from the short rate equation

- Verify from the Euler equation

$$\begin{aligned} P_{n+1,t} &= E_t[\exp(m_{t+1} + p_{n,t+1})] \\ &= E_t[\exp(-y_{1,t} - \frac{1}{2}\lambda^2 x_t \sigma^2 - \lambda\sqrt{x_t}\varepsilon_{t+1} + A_n + B_n x_{t+1})] \\ &= \exp\left(A_n + B_n\delta(1-\rho) + (B_n\rho - 1 - \lambda B_n \sigma^2 + \frac{1}{2}B_n^2\sigma^2)x_t\right) \end{aligned}$$

Cox-Ingersoll-Ross: Bond Prices

- Solution for log price in CIR model for $A_0 = B_0 = 0$:

$$A_{n+1} = A_n + B_n \delta(1 - \rho)$$
$$B_{n+1} = -1 + B_n \rho + \frac{B_n^2 \sigma^2}{2} - \lambda B_n \sigma^2$$

- Excess return on n-period bond:

$$\begin{aligned} r_{n+1,t+1} - y_{1,t} &= p_{n,t+1} - p_{n+1,t} - y_{1,t} \\ &= -\frac{B_n^2 \sigma^2}{2} x_t + \lambda B_n x_t \sigma^2 + B_n \sqrt{x_t} \varepsilon_{t+1} \end{aligned}$$

- Decompose excess return into:
  1. Jensen term
  2. Bond risk premium: positive if $\lambda < 0$, time-varying
  3. Unexpected bond return: heteroskedastic

Cox-Ingersoll-Ross: Properties

- One-factor model that can generate upward and downward sloping term structures
- Interest rates can no longer become negative (in continuous time limit)
- Yields and bond returns are heteroskedastic
- All bond yields are perfectly correlated, inherit autocorrelation of short rate
- Positive and time-varying bond risk premium, driven by same factor that drives yields

Affine Term Structure vs. C-CAPM

- Can write C-CAPM SDF in similar form:

$$M_{t+1} = \exp\left\{\log \beta - \gamma \log \Delta c_{t+1}\right\}$$

- If $\log \Delta c_{t+1} \overset{iid}{\sim} N(\mu, \sigma^2)$ then:

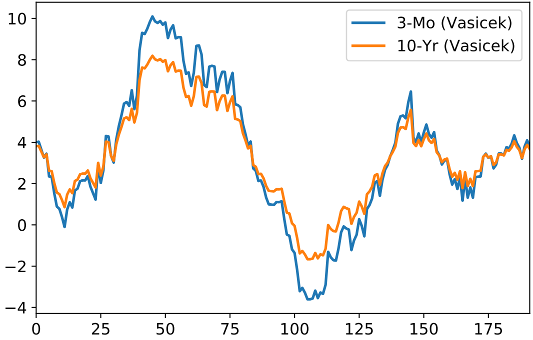
$$y_{1,t} = y_{n,t} = -\log \beta + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2$$

- If mean consumption growth is positively autocorrelated (e.g., $\mu_t = x_t$, long run risk): $\gamma > 0$ implies long bonds are hedges, negative risk premium.

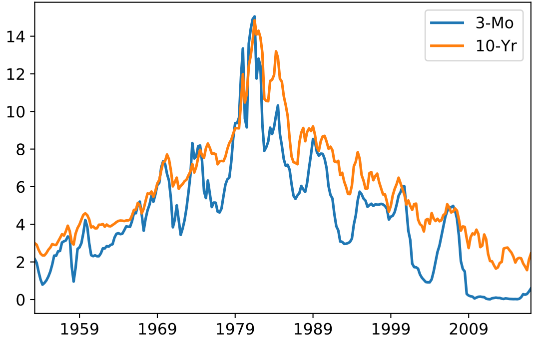
- Positive risk premium requires negatively autocorrelated consumption growth or change to specification (e.g., stochastic inflation).

Vasicek Model vs. Data

- Vasicek calibration: $\rho = 0.982$, $\sigma = 0.0057$, $\lambda = -1.4$ (all quarterly):



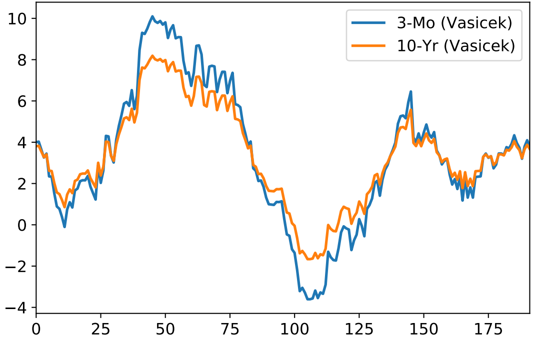
(a) Vasicek Model



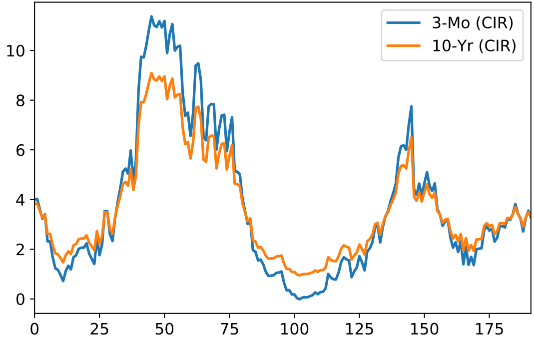
(b) Data

Vasicek Model vs. CIR Model

- CIR calibration: $\rho = 0.982$, $\sigma = 0.0304$, $\lambda = -1.4$ (all quarterly).
  - Same shocks applied to both models.



(a) Vasicek Model



(b) CIR Model

Model 4: Affine Term Structure Model (ATSM)

- Standard $k-$factor essentially affine model in discrete time features (generalizes Vasicek):
  1. Multiple risk factors driving yields
  2. Time-varying risk premia
- Suppose we have k factors: $x_t \in R^k$:

$$x_{t+1} = \Gamma x_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0, \Sigma)$$

- These factors are latent, but we can recover them from yields

Model 4: ATSM

- SDF:

$$M_{t+1} = \exp\left(-y_{1,t} - \frac{1}{2}\lambda_t'\Sigma\lambda_t - \lambda_t'\varepsilon_{t+1}\right)$$

- To keep the model affine:

$$\begin{aligned} y_{1,t} &= \delta_0 + \delta_x' x_t, \\ \lambda_t &= \Lambda_0 + \Lambda_x x_t \end{aligned}$$

- Identification: Dai and Singleton (2000)

ATSM: Bond Prices

- Solving the model is the same as before:

$$
\begin{aligned}
P_{1,t} &= \exp\left(-\delta_0 - \delta_x' x_t\right) \\
&= \exp\left(A_1 + B_1' x_t\right),
\end{aligned}
$$

  for $A_1 = -\delta_0,\ B_1' = -\delta_x'$.

- For any n :

$$
\begin{aligned}
P_{n,t} &= E_t\left(\exp\left(-y_{1,t} - \frac{1}{2}\lambda_t'\Sigma\lambda_t - \lambda_t'\varepsilon_{t+1} + A_{n-1} + B_{n-1}' x_{t+1}\right)\right) \\
&= \exp\left(-y_{1,t} + \frac{1}{2}B_{n-1}'\Sigma B_{n-1} - \lambda_t'\Sigma B_{n-1} + A_{n-1} + B_{n-1}'\Gamma x_t\right) \\
&= \exp\left(A_n + B_n' x_t\right),
\end{aligned}
$$

  where

$$
\begin{aligned}
A_n &= -\delta_0 + A_{n-1} + \frac{1}{2}B_{n-1}'\Sigma B_{n-1} - \Lambda_0'\Sigma B_{n-1}, \\
B_n' &= -\delta_x' + B_{n-1}'\left(\Gamma - \Sigma\Lambda_x\right)
\end{aligned}
$$

ATSM: Bond Prices

- Solving the model is the same as before:

$$\begin{aligned} P_{1,t} &= \exp\left(-\delta_0 - \delta_x' x_t\right) \\ &= \exp\left(A_1 + B_1' x_t\right), \end{aligned}$$

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- For any n :

$$\begin{aligned} P_{n,t} &= E_t\left(\exp\left(-y_{1,t} - \frac{1}{2}\lambda_t'\Sigma\lambda_t - \lambda_t'\varepsilon_{t+1} + A_{n-1} + B_{n-1}' x_{t+1}\right)\right) \\ &= \exp\left(-y_{1,t} + \frac{1}{2}B_{n-1}'\Sigma B_{n-1} - \lambda_t'\Sigma B_{n-1} + A_{n-1} + B_{n-1}'\Gamma x_t\right) \\ &= \exp\left(A_n + B_n' x_t\right), \end{aligned}$$

  where

$$\begin{aligned} A_n &= -\delta_0 + A_{n-1} + \frac{1}{2}B_{n-1}'\Sigma B_{n-1} - \Lambda_0'\Sigma B_{n-1}, \\ B_n' &= -\delta_x' + B_{n-1}'\left(\Gamma - \Sigma\Lambda_x\right) \end{aligned}$$

ATSM: Bond Returns

- Bond returns

$$
\begin{aligned}
r_{n,t+1} &= p_{n-1,t+1} - p_{n,t} \\
&= A_{n-1} - A_n + B'_{n-1}\Gamma x_t - B'_n x_t + B'_{n-1}\varepsilon_{t+1} \\
&= \delta_0 - \frac{1}{2}B'_{n-1}\Sigma B_{n-1} + \Lambda'_0 \Sigma B_{n-1} + \delta'_x x_t + B'_{n-1}\Sigma\Lambda_x x_t + B'_{n-1}\varepsilon_{t+1} \\
&= y_{1,t} - \frac{1}{2}B'_{n-1}\Sigma B_{n-1} + B'_{n-1}\Sigma\lambda_t + B'_{n-1}\varepsilon_{t+1}
\end{aligned}
$$

- Components:
  1. Short rate
  2. Jensen term
  3. Bond risk premium (time-varying)
  4. Unexpected bond return

ATSM: Bond Risk Premium

- The bond risk premium can also be computed directly:

$$\log E_t \left[\exp\left(m_{t+1} + r_{n,t+1}\right)\right] = 0,$$

which implies:

$$E_t\left(r_{n,t+1}\right) - y_{1,t} + \frac{1}{2}\mathrm{Var}_t\left(r_{n,t+1}\right) + \mathrm{Cov}_t\left(m_{t+1}, r_{n,t+1}\right) = 0$$

The bond risk premium is given by:

$$-\mathrm{Cov}_t\left[m_{t+1}, r_{n,t+1}\right] = B_{n-1}' \Sigma \lambda_t$$

Inverting Yields to Recover Factors

- We started with k latent factors
- However, we can "invert yields" to recover the factors
- Pick any k yields:

$$\begin{bmatrix} A_{n_1} \\ \dots \\ A_{n_k} \end{bmatrix} + \begin{bmatrix} B'_{n_1} \\ \dots \\ B'_{n_k} \end{bmatrix} x_t = -\frac{1}{n} \begin{bmatrix} y_{n_1,t} \\ \dots \\ y_{n_k,t} \end{bmatrix},$$

- Could invert to express all the factors as affine functions of the yields
  - This is important when you have observable macro factors!
  - In the data, macro variables are not linear combinations of bond yields (Joslin, Singleton, and Zhu, RFS 2011)

Properties

- Yield curve can take on any shape
- Risk premium is time-varying
- There is a k-factor structure in bond yields
- Interest rates can become negative
- Yields and bond returns are homoskedastic

Model 4: ATSM + Heteroskedasticity

- $k-$factor heteroskedastic Gaussian ATSM (generalizing CIR model):

$$x_{t+1} = \Gamma x_t + V(x_t)^{1/2}\varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0, I)$$

- $V(x)$ is a diagonal matrix with entries $V_{ii}(x_t) = \alpha_i + \beta_i x_t$
- The SDF is given by (constant market prices of risk)

$$M_{t+1} = \exp\left(-y_{1,t} - \frac{1}{2}\lambda' V(x_t)\lambda - \lambda' V(x_t)^{1/2}\varepsilon_{t+1}\right)$$

- See Duffie-Kan (1996)

Pricing Real Bonds

- Most term structure models formulated in nominal terms
- TIPS data available since 1997, but initially illiquid
- Once we specify an inflation process, we can price real bonds:

$$
\begin{aligned}
P^R_{n,t} &= E_t\left[M_{t+1} P^R_{n-1,t+1} \frac{\Pi_{t+1}}{\Pi_t}\right] \\
&= E_t\left[M^R_{t+1} P^R_{n-1,t+1}\right],
\end{aligned}
$$

$$
M_{t+1} \frac{\Pi_{t+1}}{\Pi_t} = M^R_{t+1}
$$

- To get closed-form solutions, we can model inflation $\pi_{t+1} = \log(\Pi_{t+1}/\Pi_t)$:

$$
\pi_{t+1} = \zeta_0 + \zeta_1' x_t + \sigma_\pi' \varepsilon_{t+1},
$$

Affine Term Structure Models of Equity

Term Structure of Equity

- Can assets other than bonds have a term premium? Yes!
  - For a given risky cash flow process, compare price of receiving realization at different horizons.
  - Gormsen and Lazarus (2023): many anomalies vanish controlling for cash flow duration.
- Equivalent of zero-coupon bond is zero-coupon equity: asset that pays dividend n periods from now.
  - Value stocks pay more dividends in the short run: more weight on short-duration equity.
  - Growth stocks pay more dividends in the long run: more weight on long-duration equity.
- Value premium $\implies$ term structure of equity is downward sloping.

Pricing Zero-Coupon Equity

- Let $P_{n,t}$ denote the price of an asset that pays D_{t+n} at time $t+n$, and nothing otherwise.
- Derive a recursion for the price-dividend ratio for equity strips.

$$
\begin{aligned}
P_{n+1,t} &= E_t\Big[M_{t+1}P_{n,t+1}\Big] \\
\frac{P_{n+1,t}}{D_t} &= E_t\left[M_{t+1}\left(\frac{P_{n,t+1}}{D_{t+1}}\right)\left(\frac{D_{t+1}}{D_t}\right)\right] \\
pd_{n+1,t} &= \log E_t \exp\Big\{m_{t+1} + pd_{n,t+1} + \Delta d_{t+1}\Big\}.
\end{aligned}
$$

- Note: this is the price of a strip paying at $t+n$ divided by dividend at time t.

Pricing Zero-Coupon Equity

- Guess the functional form $pd_{n,t} \equiv \log(P_{n,t}/D_t) = A_{n,0} + A_{n,x}' x_t$. Initialize $A_{0,0} = 0, A_{n,x}' = 0$.
- Substitute in our functional forms to obtain:

$$\begin{aligned}
pd_{n+1,t} &= \log E_t \exp\left\{-y_{1,t} - \frac{1}{2}\lambda_t' \Sigma \lambda_t - \lambda_t' \varepsilon_{t+1} + A_{n,0} + A_{n,x}' x_{t+1} + D_0 + D_x' x_t + D_\varepsilon' \varepsilon_{t+1}\right\} \\
&= \log E_t \exp\left\{-y_{1,t} - \frac{1}{2}\lambda_t' \Sigma \lambda_t + A_{n,0} + A_{n,x}' \Phi x_t + D_0 + D_x' x_t + \left(A_{n,x}' B + D_\varepsilon' - \lambda_t'\right) \varepsilon_{t+1}\right\} \\
&= -y_{1,t} - \frac{1}{2}\lambda_t' \Sigma \lambda_t + A_{n,0} + A_{n,x}' \Phi x_t + D_0 + D_x' x_t \\
&\qquad + \frac{1}{2}\left(A_{n,x}' B + D_\varepsilon' - \lambda_t'\right) \Sigma \left(A_{n,x}' B + D_\varepsilon' - \lambda_t'\right)' \\
&= -\delta_0 - \delta_x' x_t + A_{n,0} + A_{n,x}' \Phi x_t + D_0 + D_x' x_t \\
&\qquad + \frac{1}{2}\left(A_{n,x}' B + D_\varepsilon'\right) \Sigma \left(A_{n,x}' B + D_\varepsilon'\right)' - \left(A_{n,x}' B + D_\varepsilon'\right) \Sigma \Lambda_0 - \left(A_{n,x}' B + D_\varepsilon'\right) \Sigma \Lambda_x' x_t
\end{aligned}$$

Pricing Zero-Coupon Equity

- Collecting terms:

$$
\begin{aligned}
pd_{n+1,t} &= A_{n+1,0} + A'_{n+1,x} x_t \\
A_{n+1,0} &= -\delta_0 + A_{n,0} + D_0 + \frac{1}{2} \left(A'_{n,x} B + D'_{\varepsilon} \right) \Sigma \left(A'_{n,x} B + D'_{\varepsilon} \right)' - \left(A'_{n,x} B + D'_{\varepsilon} \right) \Sigma \Lambda_0 \\
A'_{n+1,x} &= -\delta'_x + A'_{n,x} \Phi + D'_x - \left(A'_{n,x} B + D'_{\varepsilon} \right) \Sigma \Lambda'_x
\end{aligned}
$$

- Very similar to bond pricing recursion with addition of *D* terms.
  - Account for cash flow news and uncertainty.

Lettau and Wachter (2007)

- Fundamental shocks $\varepsilon_{t+1} \sim N(0, I_k)$.
- Log aggregate dividend growth process:

$$\Delta d_{t+1} = z_t + \sigma_d' \varepsilon_{t+1}$$
$$z_{t+1} = (1 - \phi_z) g + \phi_z z_t + \sigma_z' \varepsilon_{t+1}.$$

- Stochastic discount factor (only dividend risk priced):

$$M_{t+1} = \exp\left\{ -r^f - \frac{1}{2}\gamma_t^2 - \gamma_t \varepsilon_{d,t+1} \right\}$$

for risk tolerance γ_t and dividend risk $\varepsilon_{d,t+1}$ defined by

$$\gamma_{t+1} = (1 - \phi_\gamma)\bar{\gamma} + \phi_\gamma \gamma_t + \sigma_\gamma' \varepsilon_{t+1}$$
$$\varepsilon_{d,t+1} = \frac{\sigma_d' \varepsilon_{t+1}}{||\sigma_d||}$$

Pricing Zero-Coupon Equity

- We can also solve by hand to gain intuition. Conjecture the functional form:

$$\frac{P_{n,t}}{D_t} = \exp\left\{A_n + B_{\gamma,n}\gamma_t + B_{z,n}z_t\right\}.$$

- Solution:

$$\begin{aligned}
B_{z,n} &= \frac{1-\phi_z^n}{1-\phi_z} \\
B_{\gamma,n} &= B_{\gamma,n-1}\left(\phi_\gamma - \sigma_\gamma' \frac{\sigma_d}{||\sigma_d||}\right) - (\sigma_d' + B_{z,n-1}\sigma_z')\frac{\sigma_d}{||\sigma_d||} \\
A_n &= -r^f + g + \frac{1}{2}\sigma_{n,t}'\sigma_{n,t} + A_{n-1} + B_{\gamma,n-1}(1-\phi_\gamma)\bar{\gamma} \\
\sigma_{n,t} &= \sigma_d + B_{z,n-1}\sigma_z + B_{\gamma,n-1}\sigma_\gamma.
\end{aligned}$$

- $pd_{n,t}$ increasing in expected div. growth z_t, decreasing in risk price γ_t for $\sigma_\gamma'\sigma_d$ and $\sigma_z'\sigma_d \simeq 0$.

Market Returns

- Holding period return to zero-coupon equity:

$$r_{n,t+1} - r^f - \frac{1}{2}\sigma'_{n,t}\sigma_{n,t} = \sigma'_{n,t}\frac{\sigma'_d}{||\sigma_d||}\gamma_t$$

- Aggregate market *PD* ratio and market returns:

$$\frac{P_t^m}{D_t} = \sum_{n=1}^{\infty}\frac{P_{n,t}}{D_t}$$

$$R_{t+1}^m = \frac{(P_{t+1}^m/D_{t+1}) + 1}{P_t^m/D_t}\frac{D_{t+1}}{D_t}$$

- Price of firm that pays deterministic shares $s_{j,t}$ of aggregate dividend:

Firm Returns

- Price of firm that pays deterministic shares $s_{j,t}$ of aggregate dividend:

$$P^F_{j,t} = \sum_{n=1}^{\infty} s_{j,t+n} P_{n,t}$$

- Assume N firms who rotate through the same deterministic sequence $(\bar{s}_1, \ldots, \bar{s}_N)$ so that

$$s_{j,t} = \bar{s}_{(t \bmod N)+j}$$

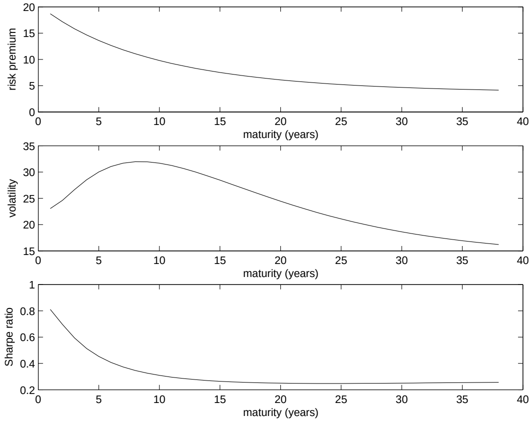
- Define shares so that

$$\bar{s}_{i+1} = \gamma \left(1 - \frac{i-1}{N/2-1}\right) \bar{s}_i$$

Results

- Reproduces asset pricing moments, value premium, downward sloping equity term structure.

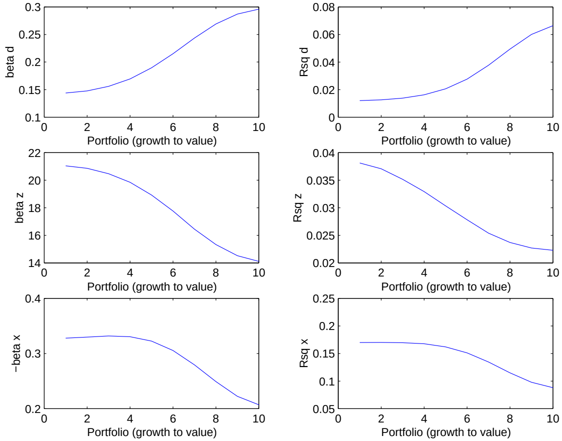
Figure 3: Properties of Zero-Coupon Equity



Results

- Value stocks more exposed to ε_d, less exposed to ε_z, ε_γ.

Figure 8: Regressions on Fundamental Shocks



Results

- How are these risks priced?
- Dividend risk ε_d has positive price:
  - Investors have high MU when dividends low.
  - Value stocks more exposed to this risk.
- Dividend growth risk ε_z has negative price:
  - Calibration: ε_z negatively correlated with ε_d.
  - Stocks that load on z_t (e.g., growth stocks) are hedges.
- Sentiment risk ε_γ is not priced.
  - Changes in risk tolerance uncorrelated with fundamentals.
- Result: value stocks have higher risk premium!

Lettau and Wachter (2011)

- Why is the term structure of equity downward sloping while the term structure of bonds is upward sloping?
- Extend model to add inflation process

$$\Delta \pi_{t+1} = q_t + \sigma_\pi' \varepsilon_{t+1}$$
$$q_{t+1} = (1 - \phi_q)\bar{q} + \phi_q q_t + \sigma_q' \varepsilon_{t+1}$$

and risk-free rate process

$$r^f_{t+1} = (1 - \phi_r)\bar{r}^f + \phi_r r^f_t + \sigma_r' \varepsilon_t$$
$$M_{t+1} = \exp\left\{ -r^f_{t+1} - \frac{1}{2}\gamma_t^2 - \gamma_t \varepsilon_{d,t+1} \right\}$$

Stock vs. Bond Term Structure

- Key is calibration of shock correlations.
- Negative correlation between ε_d and ε_r: yields low when dividends high $\implies$ premium for long-term real bonds.
- Negative correlation between ε_d and persistent inflation: nominal bonds risky, especially long-term ones.

Table 2: Conditional cross-correlations of shocks

Variable	$\Delta\pi_t$	z_t	q_t	r^f_{t+1}	x_t
Δd_t	-0.30	-0.83	-0.30	-0.30	0
$\Delta\pi_t$		0	1.00	0	0
z_t			0	0	0.35
q_t				0	0
r^f_{t+1}					0

General Asset Pricing Framework

Affine Term Structure Model of Equity

- Can also use this structure for a tractable model of equity.
- Let x_t be a vector of state variables. Transition equation (note: $c = (I - \Phi)\bar{x}$):

$$x_{t+1} = c + \Phi x_t + B\varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim N(0, \Sigma).$$

- Log dividend growth defined by:

$$\Delta d_{t+1} = D_0 + D_x' x_t + D_\varepsilon \varepsilon_{t+1}.$$

- Log stochastic discount factor, designed to deliver risk-free rate $y_{1,t}$ each period:

$$\begin{aligned} m_{t+1} &= -y_{1,t} - \frac{1}{2}\lambda_t' \Sigma \lambda_t - \lambda_t' \varepsilon_{t+1} \\ y_{1,t} &= \delta_0 + \delta_x' x_t \\ \lambda_t &= \Lambda_0 + \Lambda_x x_t. \end{aligned}$$

Affine Term Structure Model of Equity

- Guess and verify the solution for the log price-dividend ratio:

$$pd_t = A_0 + A_x' x_t.$$

- Apply the approximation

$$\log(PD_{t+1} + 1) \simeq \kappa_0 + \kappa_1 pd_{t+1}$$

where the coefficients κ_0 and κ_1 are defined by

$$\kappa_1 \equiv \frac{\overline{PD}}{\overline{PD} + 1}, \qquad \kappa_0 = \log\left(\overline{PD} + 1\right) - \kappa_1 \overline{pd}.$$

Then log return is

$$\begin{aligned} r_{t+1} &= \log\left(\frac{PD_{t+1} + 1}{PD_t} \cdot \frac{D_{t+1}}{D_t}\right) = \Delta d_{t+1} + \log(PD_{t+1} + 1) - pd_t \\ &\simeq \Delta d_{t+1} + \kappa_0 + \kappa_1 pd_{t+1} - pd_t \end{aligned}$$

Affine Term Structure Model of Equity

- Apply our technology for Δd_{t+1} and law of motion for x_t to obtain

$$
\begin{aligned}
r_{t+1} &\simeq \Delta d_{t+1} + \kappa_0 + \kappa_1 pd_{t+1} - pd_t \\
&= D_0 + D_x' x_t + D_\varepsilon' \varepsilon_{t+1} + \kappa_0 + \kappa_1 (A_0 + A_x' x_{t+1}) - A_0 - A_x' x_t \\
&= D_0 + D_x' x_t + D_\varepsilon' \varepsilon_{t+1} + \kappa_0 + \kappa_1 \Big(A_0 + A_x'(c + \Phi x_t + B\varepsilon_{t+1})\Big) - A_0 - A_x' x_t \\
&= \underbrace{D_0 + \kappa_0 - (1-\kappa_1)A_0 + \kappa_1 A_x' c}_{R_0} + \underbrace{\Big(D_x' - A_x'(I - \kappa_1 \Phi)\Big)}_{R_x'} x_t + \underbrace{\Big(D_\varepsilon' + \kappa_1 A_x' B\Big)}_{R_\varepsilon'} \varepsilon_{t+1} \\
&= R_0 + R_x' x_t + R_\varepsilon' \varepsilon_{t+1}
\end{aligned}
$$

where

$$
\begin{aligned}
R_0 &\equiv D_0 + \kappa_0 - (1-\kappa_1)A_0 \\
R_x' &\equiv D_x' - A_x'(I - \kappa_1 \Phi) \\
R_\varepsilon' &\equiv D_\varepsilon' + \kappa_1 A_x' B
\end{aligned}
$$

Affine Term Structure Model of Equity

- Substituting, we obtain

$$m_{t+1} + r_{t+1} = (R_0 - \delta_0) + (R_x' - \delta_x')x_t - \frac{1}{2}\lambda_t'\Sigma\lambda_t + (R_\varepsilon - \lambda_t)'\varepsilon_{t+1}.$$

- Applying $E_t[M_{t+1}R_{t+1}] = 1$ in the lognormal setting, we have

$$\begin{aligned}
0 &= \log E_t[M_{t+1}R_{t+1}] = E_t[m_{t+1}] + E_t[r_{t+1}] + \mathrm{Var}_t(m_{t+1} + r_{t+1}) \\
&= (R_0 - \delta_0) + (R_x' - \delta_x')x_t - \frac{1}{2}\lambda_t'\Sigma\lambda_t + \frac{1}{2}(R_\varepsilon - \lambda_t)'\Sigma(R_\varepsilon - \lambda_t) \\
&= (R_0 - \delta_0) + (R_x' - \delta_x')x_t + \frac{1}{2}R_\varepsilon'\Sigma R_\varepsilon' - R_\varepsilon'\Sigma\lambda_t \\
&= (R_0 - \delta_0) + (R_x' - \delta_x')x_t + \frac{1}{2}R_\varepsilon'\Sigma R_\varepsilon' - R_\varepsilon'\Sigma\Lambda_0 - R_\varepsilon'\Sigma\Lambda_x x_t \\
&= \left(R_0 - \delta_0 + \frac{1}{2}R_\varepsilon'\Sigma R_\varepsilon' - R_\varepsilon'\Sigma\Lambda_0\right) + \left(R_x' - \delta_x' - R_\varepsilon'\Sigma\Lambda_x\right)x_t.
\end{aligned}$$

Affine Term Structure Model of Equity

- By the method of undetermined coefficients, we obtain

$$0 = R_0 - \delta_0 - R'_\varepsilon \Sigma \Lambda_0 + \frac{1}{2} R'_\varepsilon \Sigma R'_\varepsilon \tag{1}$$

$$0 = R'_x - \delta'_x - R'_\varepsilon \Sigma \Lambda_x. \tag{2}$$

- Substituting our definitions into (2) yields

$$\begin{aligned} 0 &= R'_x - \delta'_x - R'_\varepsilon \Sigma \Lambda_x \\ 0 &= D'_x - A'_x (I - \kappa_1 \Phi) - \delta'_x - \left(D'_\varepsilon + \kappa_1 A'_x B\right) \Sigma \Lambda_x \\ A'_x &= \left(D'_x - \delta'_x - D'_\varepsilon \Sigma \Lambda_x\right) \left[(I - \kappa_1 \Phi) + \kappa_1 B \Sigma \Lambda_x\right]^{-1}. \end{aligned}$$

- Given solution for A'_x, can solve (1) for A_0 using nonlinear equation solver.
  - Keep in mind that A_0 affects κ_0 and κ_1.

Affine Term Structure Model of Equity

- This is a great framework for many applications.
- Model is **general**, can handle:
  - Multiple factors with VAR structure.
  - Time varying risk-free rate.
  - Time varying prices of risk.
- Model is **tractable**:
  - Semi-closed form, only requires solving single nonlinear equation.
  - Linear-Gaussian structure for exogenous states + price-dividend ratios.
  - Suitable for estimation using Kalman filter (alternative: VAR + restrictions).

Case Study: “How the Wealth Was Won” by Greenwald, Lettau, and Ludvigson

Motivation

- Stock market growth has been highly variable, even at long horizons.
- Nearly 5x higher growth from 1989 - 2017 compared to previous 1966 - 1988 period. Why?
- Textbook theory: long-horizon changes should be driven by economic growth.

Avg. Real Growth	1966 - 1988	1989 - 2017
Market Equity	1.6%	7.5%

Source: NIPA, Flow of Funds

Motivation

- Surprisingly, growth of corporate output yields the opposite pattern.
- Output grew $\sim$ 50% faster during low-return vs. high-return period.
- Are cash flows irrelevant for market valuation?

Avg. Real Growth	1966 - 1988	1989 - 2017
Market Equity	1.6%	7.5%
Corporate Output	3.9%	2.6%

Source: NIPA, Flow of Funds

Motivation

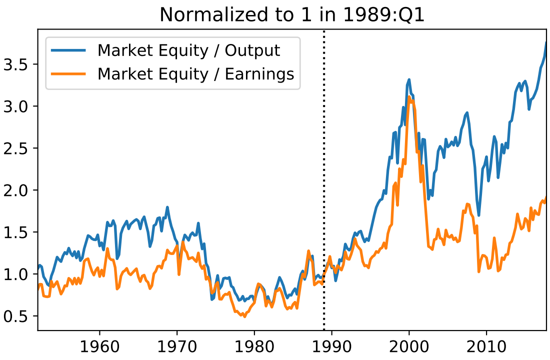
- In this paper, we propose an alternative resolution.
- Equity is claim on firm earnings, small and volatile share of firm output.
- Earnings growth was nearly 3x larger in 1989 - 2017 period relative to 1966 - 1988.

Avg. Real Growth	1966 - 1988	1989 - 2017
Market Equity	1.6%	7.5%
Corporate Output	3.9%	2.6%
Corporate Earnings	1.8%	5.1%

Source: NIPA, Flow of Funds

Valuation Ratios

- Shifts in earnings shares $\Longrightarrow$ divergence between stock market and broader economy.
- Market Equity/Output ratio ends sample (2017:Q4) at a post-war high.
- Market Equity/Earnings does not, well below historical peak.



This Paper

- **Question:** which fundamental forces have driven the post-war value of market equity?
- **Approach:** estimate flexible structural model of the equity market.
  - Fundamentals: factor shares, risk aversion, risk-free rates, and economic growth.
  - Decompose influence on value of market equity quarter-by-quarter.
- **Main Finding:** changes in earnings share powerful driver of market equity.
  - Explains **44%** of ME growth from 1989 - 2017, compared to **25%** created by economic growth.
  - Secular changes in risk tolerance explain **18%**, while falling risk-free rates explain only **14%**.
  - Contrast to 1952 - 1988 period: ME growth lower, completely explained by **economic growth**.
  - Growth in earnings share over 1989 - 2017 period overwhelmingly at expense of **labor share**.
  - Favorable shock realizations added **2.1ppt** to mean log excess return over 1952 - 2017.

Productive Technology

- Production function: $Y_t = A_t N_t^{\alpha} K_t^{1-\alpha}$.
  - TFP process A_t is random walk: $\Delta \log A_t = \varepsilon_{a,t}, \quad \varepsilon_{a,t} \sim N(0, \sigma_a^2)$.
  - Balanced growth: capital K_t, efficiency units of labor N_t, grow deterministically at rate g.
- Factor shares:
  - Time-varying share S_t of output accrues to shareholder as earnings: $E_t = S_t Y_t$.
  - Remaining share $1 - S_t$ of output paid to labor compensation, taxes, interest, etc.
- Investment:
  - Capital growth requires reinvesting fraction ω of corporate output.
  - Remaining cash flows C_t paid out, consumed by shareholders: $C_t = (S_t - \omega) Y_t$.
  - Log-linear approximation: $c_t = \xi s_t + y_t, \quad \xi \equiv S/(S - \omega)$.
  - Average value of S/C in the data implies $\xi = 2.00$ (leverage effect). ▸ More

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  - Average value of S/C in the data implies $\xi = 2.00$ (leverage effect). More

Preferences

- Rep. shareholder has log-affine stochastic discount factor over cash flows C_t:

$$M_{t+1} = \beta_t \left(\frac{C_{t+1}}{C_t} \right)^{-x_t}$$

- Time-varying discount factor β_t:

$$\beta_t = \frac{\exp(-\delta_t)}{E_t \exp(-x_t \Delta c_{t+1})}$$

  ensures that the risk-free rate equals δ_t at all times.

- Time-varying risk price x_t provides secular variation in risk premia.

- Second-order SDF accounts for higher risk when profit share low (leverage risk effect):

$$\log M_{t+1} \simeq -\delta_t - \mu_t - x_t \Delta c_{t+1} + \underbrace{\bar{x}\xi(1-\xi)\left(E_t[s_{t+1}] - \bar{s}\right)\Delta s_{t+1}}_{\text{leverage risk effect}}$$

Leverage Effect Intuition

- Cash flow share of output equal to earnings share (S_t) net of reinvestment share (ω).
  - Average earnings share $S_t \simeq 12\%$, reinvestment $\omega \simeq 6\%$, so average cash flow share $\simeq 6\%$.
- Leverage effect: $S_t \uparrow 50\%$ (12% to 18%) increases cash flow share by 100% (6% to 12%).
- Leverage risk effect: this effect is stronger as S_t gets closer to ω.
  - $S_t \uparrow 25\%$ (8% to 10%) increases cash flow share by 100% (2% to 4%).
  - $S_t \uparrow 25\%$ (16% to 20%) increases cash flow share by 40% (10% to 14%).
- Very similar to external habit mechanism of Campbell and Cochrane (1999) but based on earnings, reinvestment rather than consumption, habit.

Parameterization

- Model each state variable as sum of low and high frequency component:

$$s_t = \bar{s} + s_{LF,t} + s_{HF,t}$$

$$\delta_t = \bar{\delta} + \delta_{LF,t} + \delta_{HF,t}$$

$$x_t = \bar{x} + x_{LF,t} + x_{HF,t}$$

- Model each component as independent Gaussian AR(1) with mean zero:

$$s_{LF,t} = \phi_{s,LF}\, s_{LF,t-1} + \varepsilon_{s,LF,t} \qquad s_{HF,t} = \phi_{s,HF}\, s_{HF,t-1} + \varepsilon_{s,HF,t}$$

$$\delta_{LF,t} = \phi_{\delta,LF}\, \delta_{LF,t-1} + \varepsilon_{\delta,LF,t} \qquad \delta_{HF,t} = \phi_{\delta,HF}\, \delta_{HF,t-1} + \varepsilon_{\delta,HF,t}$$

$$x_{LF,t} = \phi_{x,LF}\, x_{LF,t-1} + \varepsilon_{x,LF,t} \qquad x_{HF,t} = \phi_{x,HF}\, x_{HF,t-1} + \varepsilon_{x,HF,t}$$

Solution

- Approximating the return process as in Campbell and Shiller (1989) as

$$r_{t+1} = \kappa_0 + \kappa_1 pc_{t+1} - pc_t + \Delta c_{t+1}$$

allows for the closed form solution

$$\begin{aligned}
pc_t &= A_0 + \mathbf{A}_s' \widetilde{\mathbf{s}}_t + \mathbf{A}_\delta' \widetilde{\delta}_t + \mathbf{A}_x' \widetilde{\mathbf{x}}_t \\
\mathbf{A}_s' &= -\xi \left[\mathbf{1}' (\mathbf{I} - \mathbf{\Phi}_s) - (\mathbf{1}' \mathbf{\Sigma}_s \mathbf{1}) \mathbf{\Gamma}' \right] \left[(\mathbf{I} - \kappa_1 \mathbf{\Phi}_s) - \kappa_1 \xi \mathbf{\Sigma}_s \mathbf{1} \mathbf{\Gamma}' \right]^{-1} \\
\mathbf{A}_x' &= -\left[\left(\xi^2 (\mathbf{1}' \mathbf{\Sigma}_s \mathbf{1}) + \sigma_a^2 + \kappa_1 \xi (\mathbf{A}_s' \mathbf{\Sigma}_s \mathbf{1}) \right) \mathbf{1}' \right] (\mathbf{I} - \kappa_1 \mathbf{\Phi}_x)^{-1} \\
\mathbf{A}_\delta' &= -\mathbf{1}' (\mathbf{I} - \kappa_1 \mathbf{\Phi}_\delta)^{-1}
\end{aligned}$$

where the term ξ captures the **leverage effect**, and the term

$$\mathbf{\Gamma}' = \bar{x} \xi (\xi - 1) \mathbf{1}' \Phi$$

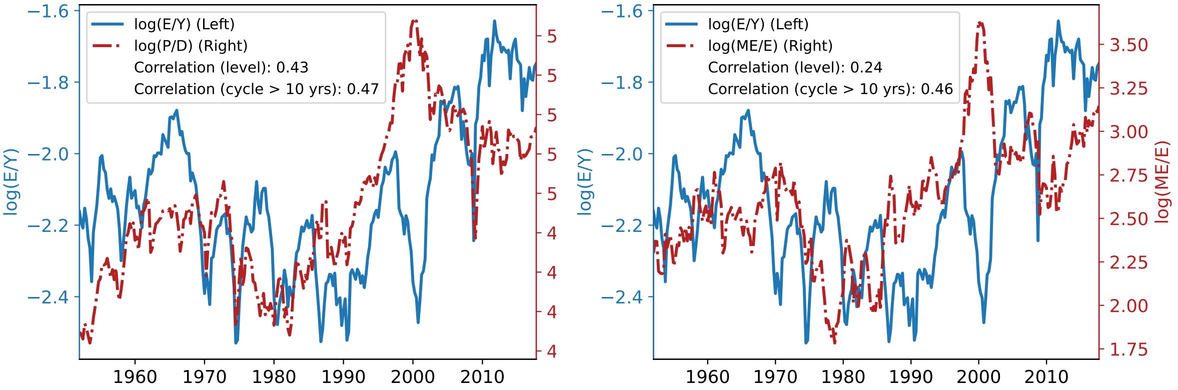
captures the influence of the **leverage risk effect**.

Solution

- With no leverage risk effect ($\Gamma = 0$), model would unambiguously predict that earnings share and *pc* ratio are negatively correlated.
- In the data, correlation is positive, supporting leverage risk effect.

(a) vs. Price-Dividend Ratio

(b) vs. Price-Earnings Ratio



Data

- Core data for US corporate sector from NIPA and Flow of Funds.
  - Single consistent source for output, earnings/labor share, and equity values.
  - "Unambiguous" capital vs. labor income (Koh, Santaeulalia-Llopis and Zheng, 2020)
- Output: real per-capita corporate domestic net value added.
- Earnings share: after-tax corporate domestic profits + foreign profits.
  - Domestic profits = domestic NVA net of taxes and interest payments.
  - Foreign profits = income on equity from foreign direct investment from BEA.
  - Not available prior to 1982, use fitted value based on total foreign equity income. More
- Stock market value = market value of corporate equities.
  - Includes both public and private US corporations.
  - Public equity makes up 80% of total value on average.

Data

- Real risk-free rate: 3-month T-Bill net of expected inflation.
  - Use ARMA(1, 1) model of inflation with GDP deflator as price index.
- Survey of Professional Forecasters 10Y forecast of average real short rates.
  - Difference of 10Y T-Bill forecast and 10Y CPI inflation forecast.
- Risk premium from SVIX-based measure Martin (2017 QJE)
- Corporate payouts: net dividends minus net equity issuance in Flow of Funds.
  - Not directly used in estimation.

Estimation

- Use six observable series over sample 1952:Q1 - 2017:Q4:

  1. Log corp. output growth: Δy_t
  2. Log corp. earnings share: s_t
  3. Log real risk-free rate: δ_t
  4. Log 10-year real short rate forecast: $\bar{r}^{40}_{f,t}$.
  5. Log corp. equity to corp. output ratio: $p_t - y_t$
  6. Risk premium implied by Martin ('17): rp_t.

- Estimate state space model using Bayesian MCMC with flat priors

  - Pin down ξ, average earnings share $\bar{s}$, risk-free rate $\bar{\delta}$, growth g, directly from data.
  - Estimate remaining 15 parameters (persistences, volatilities, average risk price, forecast bias).

Parameter Estimates

- Parameter estimates, reported at quarterly frequency.

Variable	Symbol	Mode	5%	Median	95%
Risk Price Mean	$\bar{x}$	5.8676	4.7683	6.0466	7.6212
Risk Price (HF) Pers.	$\phi_{x,HF}$	0.6781	0.5451	0.6943	0.7986
Risk Price (HF) Vol.	$\sigma_{x,HF}$	2.1724	1.6106	2.1854	2.9669
Risk Price (LF) Pers.	$\phi_{x,LF}$	0.9886	0.9809	0.9882	0.9946
Risk Price (LF) Vol.	$\sigma_{x,LF}$	0.6295	0.3736	0.6232	0.9823
Risk-Free (HF) Pers.	$\phi_{\delta,HF}$	0.8413	0.7704	0.8473	0.8938
Risk-Free (HF) Vol.	$\sigma_{\delta,HF}$	0.0017	0.0015	0.0017	0.0019
Risk-Free (LF) Pers.	$\phi_{\delta,LF}$	0.9630	0.9514	0.9639	0.9790
Risk-Free (LF) Vol.	$\sigma_{\delta,LF}$	0.0010	0.0007	0.0010	0.0013
Factor Share (HF) Pers.	$\phi_{s,HF}$	0.9035	0.8334	0.8846	0.9194
Factor Share (HF) Vol.	$\sigma_{s,HF}$	0.0527	0.0472	0.0530	0.0577
Factor Share (LF) Pers.	$\phi_{s,LF}$	0.9873	0.9780	0.9930	0.9988
Factor Share (LF) Vol.	$\sigma_{s,LF}$	0.0168	0.0084	0.0152	0.0264
Productivity Vol.	σ_a	0.0154	0.0142	0.0152	0.0164
Forecast Mean Adjustment	ν	0.0020	-0.0006	0.0018	0.0042

Parameter Estimates

- Earnings share, risk price processes highly persistent, risk-free rate less so.

Variable	Symbol	Mode	5%	Median	95%
Risk Price Mean	$\bar{x}$	5.8676	4.7683	6.0466	7.6212
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Parameter Estimates

- SDF based on cash flows rather than consumption able to match equity premium with low average risk price.

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Risk Price (HF) Pers.	$\phi_{x,HF}$	0.6781	0.5451	0.6943	0.7986
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Forecast Mean Adjustment	ν	0.0020	-0.0006	0.0018	0.0042

Parameter Estimates

- Median estimate for $A_{s,LF}$ is -0.07, implying close to zero correlation between pc_t and $s_{LF,t}$.

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Risk Price Mean	$\bar{x}$	5.8676	4.7683	6.0466	7.6212
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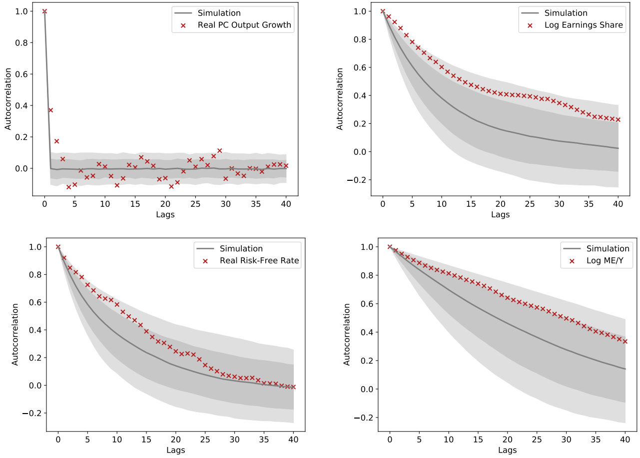
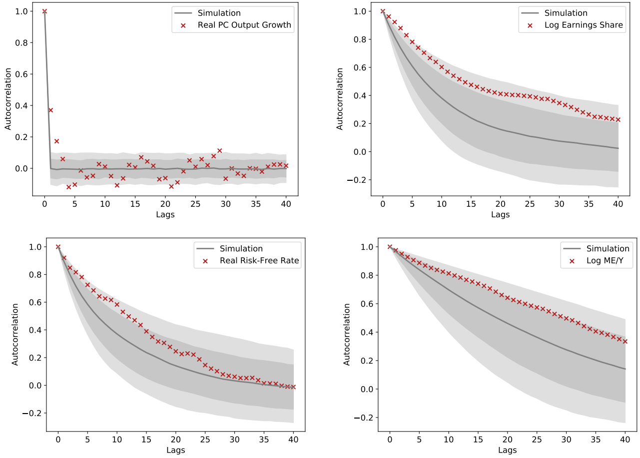
Observable Autocorrelations

- Compare sample autocorrelations in data to sample autocorrelations from 10,000 model simulations, each the length of the data sample.



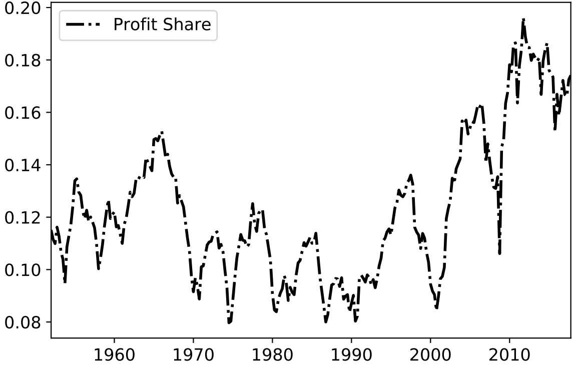
Observable Autocorrelations

- Model fits autocorrelations well, possibly understating persistence of cash flows and valuations.



Earnings Share

- Figure plots earnings share S_t.
- Highly volatile, rises from 8.9% in 1989 to 17.4% in 2017.

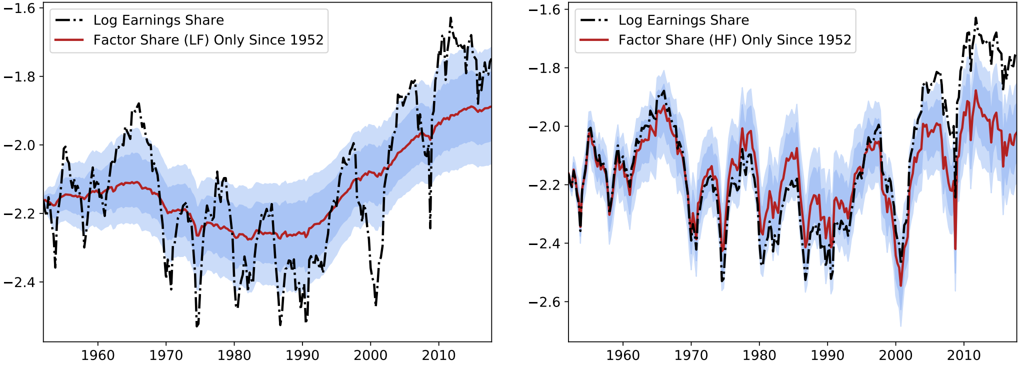


Source: NIPA.

Earnings Share: Decomposition

▸ Dividend Share

- Model decomposes earnings share into high- and low-frequency components.
- Low-frequency factor share series accurately captures trend.

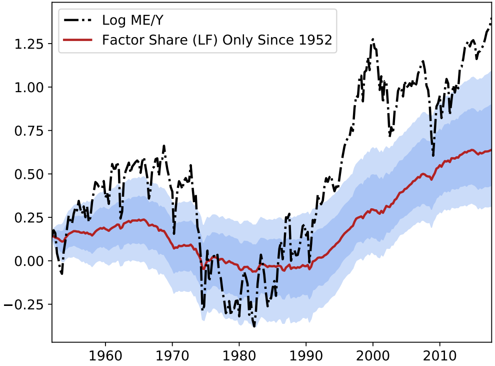


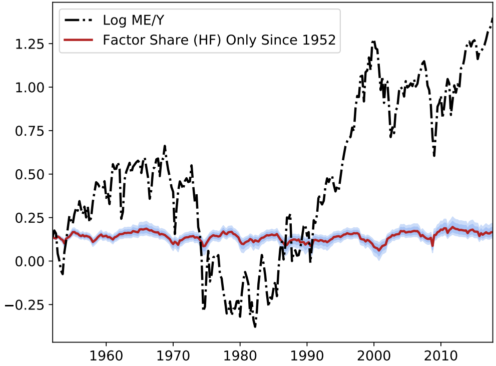
The median sample path is displayed in red, while blue shaded regions are 66.7% (darker bands) and 90% (lighter bands) credible sets. The sample spans the period 1952:Q1-2017:Q4.

Contribution of Earnings Share to Market Equity

▸ Intuition ▸ Dividend Share

- Effect on market equity depends on persistence, operating leverage (ξ), risk effect.
- Forward-looking asset prices $\implies$ LF component has much stronger effect on valuations.





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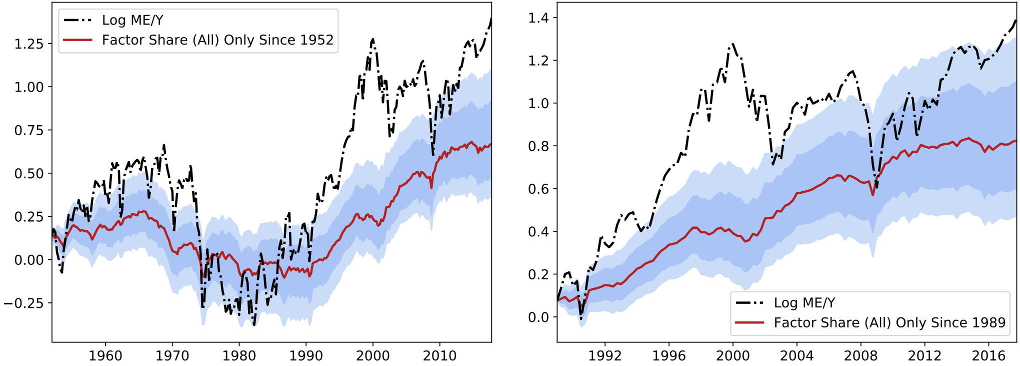
Contribution of Earnings Share to Market Equity

▸ Intuition

- Combined, shifts in factor shares drive much of slow-moving trend in ME/Y.
- Rise in earnings share explains 58% of rise from 1989 to 2017.



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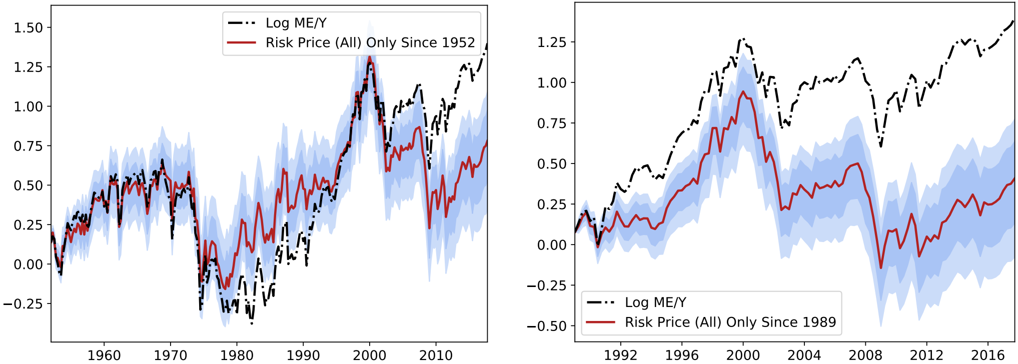
Contribution of Orthogonal Risk Price to Market Equity

- Changes in orthogonal risk price x capture most high-frequency movements.
- But miss most of the upward trend since 1989.



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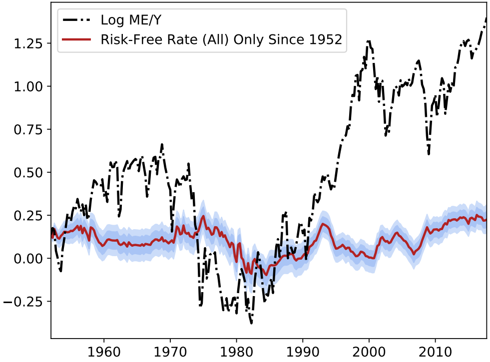
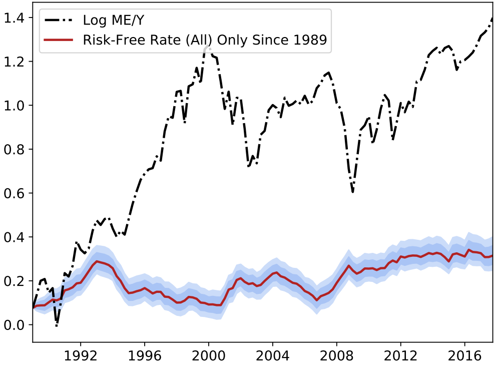
Contribution of Risk-Free Rate to Market Equity

▸ Latent States

- Changes in risk-free rates have a small impact at all horizons.
- Contrast with alternative estimates (e.g., Farhi + Gourio ('18), Corhay et al ('20)).





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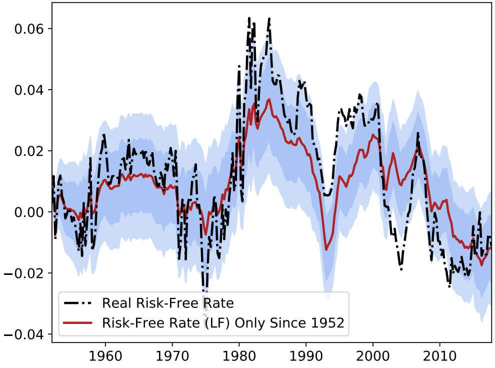
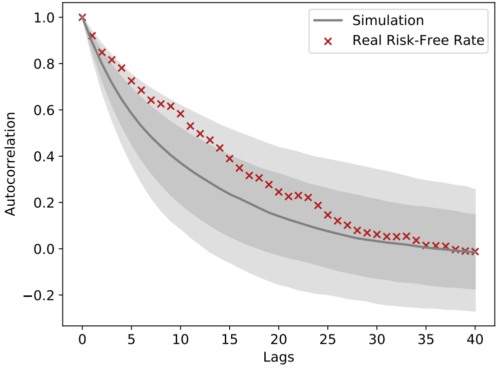
Contribution of Risk-Free Rate to Market Equity

- Key difference: we estimate persistent but non-permanent changes in risk-free rate.
- Modal estimate of LF risk-free persistence is 0.96, matching observed autocorrelations.
- Corollary: we estimate risk premia at historic lows at end of our sample. ▸ Plot





Growth Decomposition

- Changes in factor shares explain **44%** of rise since 1989, **21%** overall.
- Output explains smaller share since 1989 (**25%**), only **54%** over full sample.

	Real PC Market Equity		
Contribution	1952-2017	1952-1988	1989-2017
Total	**1405.81%**	**151.23%**	**477.34%**
Earnings Share s_t	20.50%	-21.09%	43.96%
Risk Price x_t	22.72%	25.33%	17.68%
Risk-Free Rate δ_t	3.24%	-15.65%	13.80%
Real PC Output Growth	53.54%	111.41%	24.57%

Bolded numbers in the "Total" row indicate the total growth in market equity over that period. Non-bolded numbers in the remaining rows indicate the share of the total growth in logs explained by each component, averaged over 10,000 parameter draws.

Growth Decomposition

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Growth Decomposition

- Risk price explains most of remainder.
- Nontrivial but modest role for risk-free rates.

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Bolded numbers in the "Total" row indicate the total growth in market equity over that period. Non-bolded numbers in the remaining rows indicate the share of the total growth in logs explained by each component, averaged over 10,000 parameter draws.

Growth Decomposition

- ME growth in 1952 - 1988 period overwhelmingly explained by economic growth.
- But generated less than 1/3 the growth in wealth of 1989 - 2017 period.

	Real PC Market Equity		
Contribution	1952-2017	1952-1988	1989-2017
Total	**1405.81%**	**151.23%**	**477.34%**
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Asset Pricing Moments

- **Model:** long (unconditional) simulation.
- **Fitted:** estimated (smoothed) model values fitted to this particular sample.
- **Data:** actual data values.

	Model		Fitted		Data	
Variable	Mean	StD	Mean	StD	Mean	StD
Log Equity Return	5.083	17.467	7.994	16.777	8.856	15.724
Log Excess Return	3.952	17.562	6.871	16.764	7.389	16.436
Log Price-Payout Ratio	3.683	0.404	3.317	0.361	3.437	0.462
Log Earnings Growth	2.198	8.651	2.819	11.819	2.819	11.819
Log Payout Growth	2.198	16.777	3.444	21.847	4.115	33.201

All statistics are computed for annual (continuously compounded) data. “Model” numbers are averages across 1000 simulations of the model, each the same size as our data sample, at equally spaced parameter draws. “Fitted” numbers use the estimated latent states fitted to observed data in our historical sample, averaged across the same parameter draws. The sample spans the period 1952:Q1-2017:Q4.

Asset Pricing Moments

- Payout growth higher and more volatile than earnings growth over our sample.
- Model matches pattern but understates payout growth.
- Model implied equity returns slightly lower than data as a result.

	Model		Fitted		Data	
Variable	Mean	StD	Mean	StD	Mean	StD
Log Equity Return	5.083	17.467	7.994	16.777	8.856	15.724
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Asset Pricing Moments

- Realized excess return is **2.1 percentage points** higher than unconditional average.
- Due to string of **favorable factor share shocks**, rather than ex-ante risk compensation.
- Assuming that observed returns are typical may vastly **overstate equity premia**.

	Model		Fitted		Data	
Variable	Mean	StD	Mean	StD	Mean	StD
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Summary: "How the Wealth Was Won"

- Why has the market risen over the post-war period?
- We estimate a **flexible parametric model** that allows inference from several latent components, while **inferring values** components must have taken to explain the data.
- High returns to equity due in large part to good luck, attributable to **string of shocks that reallocated rewards** from workers to shareholders.
- Realizations added **2.1 ppt** to mean excess log return, exceeding risk premium by 43%.
- Over 37 year period from 1952 - 1988, strong economic growth propelled the stock market, but created **less than one third** of the wealth generated over 29 years from 1989 to 2017.
- Key tools used: tractable term structure framework, state space tools, Bayesian (MCMC) estimation.

Conclusion

- Affine SDF models provide tractable solutions for many asset pricing models.
  - Term structure of bonds or equity.
  - General asset pricing model with time varying cash flows and risk prices.
- Semi closed-form solutions.
  - Term structure: up to recursion.
  - General assets: up to constant A_0.
- Linear structure is very well suited to estimation.
  - State space estimation using the Kalman filter.
  - VARs with restrictions: Ang and Piazzesi (2003), Lustig, Van Nieuwerburgh, Verdelhan (2013), etc.