15.472: Term Structure

Dan Greenwald

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Overview

1. Review: bond markets
2. Discrete time term structure models
    - Expectations Hypothesis Model
    - Vasicek Model
    - Cox-Ingersoll-Ross Model
    - Affine Term-Structure Model
3. Empirics
    - Sources of yield variation
    - Sources of return variation
4. Term structure of equity

Bond Prices

- Focus on zero-coupon bonds (or strips)
- $P_{n,t}$ is the price of an n-period zero coupon bond:

$$P_{n,t} = E_t\left(\prod_{i=1}^{n} M_{t+i}\right)$$

- Prices of other securities with *deterministic* cash flows follow mechanically. For instance, cash flows $\{CF_{t+i}\}$ $(i = 1, ..., n)$:

$$\sum_{i=1}^{n} P_{i,t} CF_{t+i}$$

- Log bond prices are denoted by $p_{n,t} = \log P_{n,t}$

Yields

- The n-period, continuously compounded yield is defined as:

  $$P_{n,t} = \exp\left(-y_{n,t} n\right),$$

  or:

  $$y_{n,t} = -\frac{1}{n} p_{n,t}$$

- Note: inverse relationship between prices and yields
- Current yield curve:

  `http://www.bloomberg.com/markets/rates-bonds/government-bonds/us/`

Returns and Forward Rates

- The holding-period return on an n-period bond is defined as:

$$\begin{aligned} r_{n,t+1} &= p_{n-1,t+1} - p_{n,t} \\ &= n y_{n,t} - (n-1) y_{n-1,t+1} \end{aligned}$$

- The forward rate allows you to borrow and lend at some future point in time and a pre-specified interest rate, $f_t^{n \to n+1}$

$$\begin{aligned} f_t^{n \to n+1} &= p_{n,t} - p_{n+1,t} \\ &= (n+1) y_{n+1,t} - n y_{n,t} \\ &= y_{n+1,t} + n \left(y_{n+1,t} - y_{n,t} \right) \end{aligned}$$

$\Longrightarrow$ Forward rates higher than yields if the term structure is increasing

- Note: $f_t^{0 \to 1} = y_{1,t}$

Forward rates

- We can always write:

$$\begin{aligned} p_{n,t} &= p_{n,t} - p_{n-1,t} + p_{n-1,t} - ... + p_{1,t} \\ &= -\left(f_t^{n-1\rightarrow n} + f_t^{n-2\rightarrow n-1} + \cdots + y_{1,t}\right) \end{aligned}$$

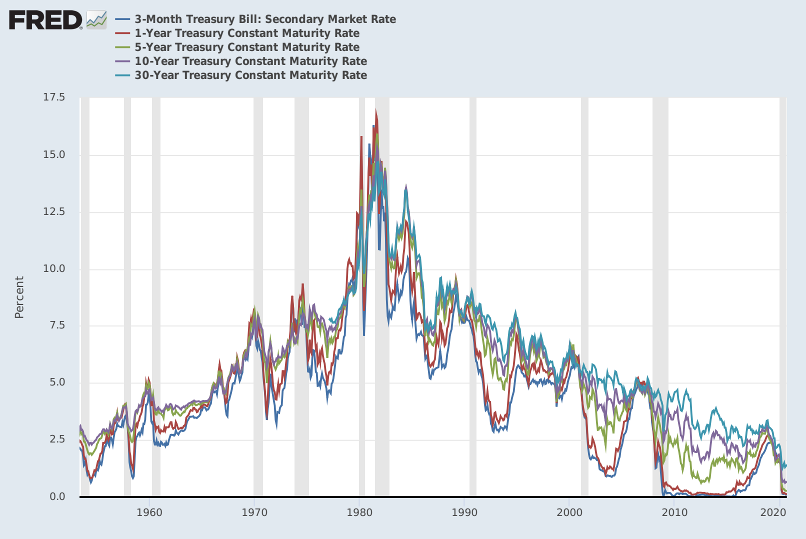
- This implies:

$$p_{n,t} = -\sum_{i=0}^{n-1} f_t^{i\rightarrow i+1}$$

or:

$$P_{n,t} = \exp\left(-\sum_{i=0}^{n-1} f_t^{i\rightarrow i+1}\right)$$

Yields on US Treasuries



Expectations Hypothesis

- Theory for understanding the *shape* of the yield curve
- Three (mathematically) equivalent statements
    1. $y_{n,t} = \frac{1}{n} E_t \left(y_{1,t} + ... + y_{1,t+n-1} \right) \; [+RP_1]$
    2. $f_t^{(n \rightarrow n+1)} = E_t \left(y_{1,t+n} \right) \; [+RP_2]$
    3. $E_t \left(p_{n-1,t+1} - p_{n,t} \right) = y_{1,t} \; [+RP_3]$
- Pure Expectations Hypothesis: $RP_1 = RP_2 = RP_3 = 0$
- Generalized Expectations Hypothesis: there is a risk premium, but it is constant over time
- PEH = risk neutrality (up to Jensen terms)

Discrete Time Term Structure Models

- We will consider four workhorse models

  1. Model based on the expectations hypothesis
  2. Vasicek model
  3. Cox-Ingersoll-Ross model
  4. $k-$factor essentially affine model in discrete time

Model 1: Expectations Hypothesis

- Pure expectations hypothesis:

$$y_{n,t} = \frac{1}{n} E_t \left\{ \sum_{j=0}^{n-1} y_{1,t+j} \right\}$$

- Stochastic process for short rate (state variable):

$$y_{1,t+1} = \delta + \rho \left(y_{1,t} - \delta \right) + \varepsilon_{t+1}$$

- Combine:

$$\begin{aligned} y_{n,t} &= \delta + \frac{1}{n} \sum_{j=0}^{n-1} E_t \left(y_{1,t+j} - \delta \right) = \delta + \frac{1}{n} \sum_{j=0}^{n-1} \rho^j \left(y_{1,t} - \delta \right) \\ &= \delta + \frac{1}{n} \frac{1 - \rho^n}{1 - \rho} \left(y_{1,t} - \delta \right) \end{aligned}$$

- What structure would deliver this outcome?

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- What structure would deliver this outcome?

Expectations Hypothesis: Properties

- One-factor model that can generate upward and downward sloping term structures
- On average, there is no slope: $E\left(y_{n,t}\right) = \delta, \ \forall n$
- All bond yields are perfectly correlated
- Interest rates can become negative
- Yields and bond returns are homoskedastic

General SDF Approach

1. State variables x_t evolve according to: $x_{t+1} = h(x_t, \varepsilon_{t+1})$
2. Define short rate as a function of the state variables: $y_{1,t} = g(x_t)$
3. The SDF takes the form
$$M_{t+1} = \frac{\exp\left(-y_{1,t} - \lambda(x_t)'\varepsilon_{t+1}\right)}{E_t \exp\left(-\lambda(x_t)'\varepsilon_{t+1}\right)}$$
4. Solve bond prices recursively using
$$P_{n+1,t} = E_t\left[M_{t+1} P_{n,t+1}\right]$$
with initial condition $P_{0,t} = 1$.

Model 2: Vasicek Model

- Similar to EH, but introduces a constant bond risk premium
- Single factor:

$$x_{t+1} = \rho x_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}\left(0, \sigma^2\right)$$

- Short rate is affine in this single factor:

$$y_{1,t} = \delta + x_t$$

- Can interpret x as level factor and ε as shocks to the level of the term structure.
- Instead of imposing the expectations hypothesis, we specify the SDF:

$$M_{t+1} = \exp\left(-y_{1,t} - \frac{1}{2}\lambda^2\sigma^2 - \lambda\varepsilon_{t+1}\right)$$

Vasicek: Bond Prices

- Recursively construct bond prices of the form

$$P_{n,t} = \exp(A_n + B_n x_t).$$

- Start with $n = 1$:

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- Start with $n = 1$:

$$P_{1,t} = \exp\left(-y_{1,t}\right) = \exp\left(A_1 + B_1 x_t\right),$$

so that

$$\begin{aligned} A_1 &= -\delta \\ B_1 &= -1. \end{aligned}$$

Vasicek: Bond Prices

- For general $n+1$:

Vasicek: Bond Prices

- For general $n+1$:

$$\begin{aligned}
P_{n+1,t} &= E_t\left(M_{t+1}P_{n,t+1}\right) \\
&= E_t\left(\exp\left(-\delta - x_t - \frac{1}{2}\lambda^2\sigma^2 - \lambda\varepsilon_{t+1} + A_n + B_n x_{t+1}\right)\right) \\
&= E_t\left(\exp\left(-\delta - x_t - \frac{1}{2}\lambda^2\sigma^2 - \lambda\varepsilon_{t+1} + A_n + B_n(\rho x_t + \varepsilon_{t+1})\right)\right) \\
&= \exp\left(-\delta - x_t + A_n + B_n\rho x_t + \frac{1}{2}B_n^2\sigma^2 - \lambda B_n\sigma^2\right) \\
&= \exp\left(A_{n+1} + B_{n+1}x_t\right),
\end{aligned}$$

with:

$$\begin{aligned}
A_n &= -\delta + A_{n-1} + \frac{1}{2}B_{n-1}^2\sigma^2 - \lambda B_{n-1}\sigma^2, \\
B_n &= -1 + B_{n-1}\rho
\end{aligned}$$

Vasicek: Bond Prices and Yields

- These difference equations can be solved recursively, and in closed-form in this simple case

$$B_n = -\frac{1-\rho^n}{1-\rho}$$

$$A_n = -n\delta + \sum_{j=1}^{n-1} \left(\frac{1}{2} B_{n-1}^2 \sigma^2 - \lambda B_{n-1} \sigma^2 \right)$$

- Note: can solve partial sums using lag operator plus indicator:

$$B_n = \mathbf{1}_{\{n>1\}} \left(\rho B_{n-1} - 1 \right)$$

$$(1 - \rho L \mathbf{1}_{\{n>1\}}) B_n = -\mathbf{1}_{\{n>1\}}$$

$$B_n = -\sum_{j=0}^{\infty} \rho^j L^j \mathbf{1}_{\{n>1\}}^{j+1} = -\sum_{j=0}^{n-1} \rho^j = -\frac{1-\rho^n}{1-\rho}$$

Vasicek: Properties

- One-factor model that can generate upward and downward sloping term structures
- Holding period return:

$$\begin{aligned} r_{n+1,t+1} &= p_{n,t+1} - p_{n+1,t} \\ &= \delta + x_t - \frac{1}{2} B_n^2 \sigma^2 + \lambda B_n \sigma^2 \end{aligned}$$

  sum of short rate, Jensen term, and risk premium.

- Risk premium is constant, positive if $\lambda < 0$

Properties

- Bond yields all perfectly correlated, given by

$$y_{n,t} = -\frac{A_n}{n} - \frac{B_n}{n} x_t$$

- Interest rates can become negative
- Yields and bond returns are homoskedastic
- Auto-correlation:
  - Because all yields and yield spreads are linear combinations of the short rate, they inherit their autocorrelations from $y_{1,t}$
  - Long yields have higher autocorrelations in the data, yield spreads have lower autocorrelations

Model 3: Cox-Ingersoll-Ross

- Single state variable follows square root process:

$$x_{t+1} = \delta + \rho\left(x_t - \delta\right) + \sqrt{x_t}\varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0, \sigma^2)$$

- The short rate is equal to this single factor:

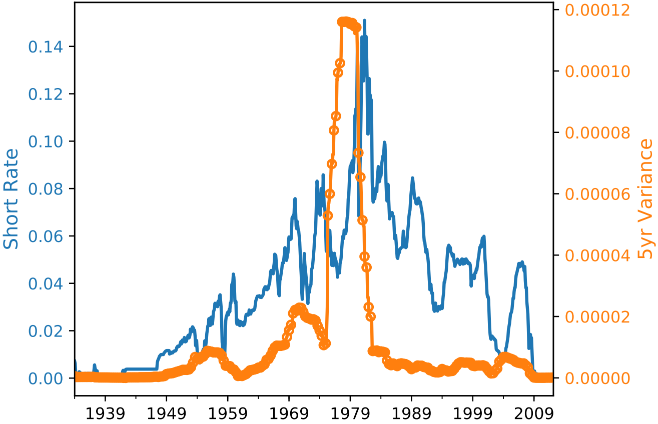
$$y_{1,t} = x_t$$

- We specify the SDF:

$$M_{t+1} = \exp\left(-y_{1,t} - \frac{1}{2}\lambda^2 x_t \sigma^2 - \lambda\sqrt{x_t}\varepsilon_{t+1}\right)$$

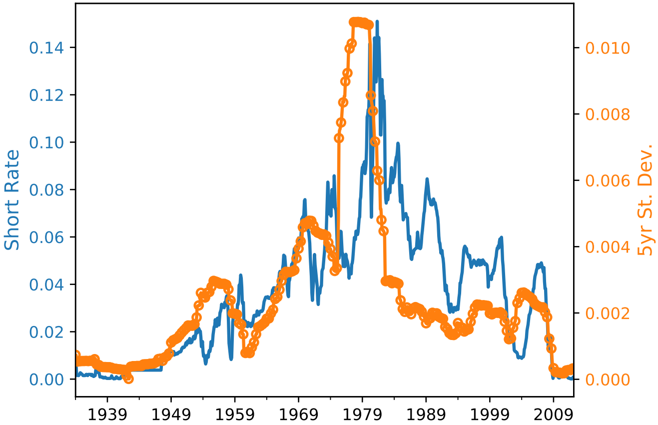
Square Root Process: Evidence

- 3mo rate vs. 5y-ahead realized variance:



Square Root Process: Evidence

- 3mo rate vs. 5y-ahead realized standard deviation:



Cox-Ingersoll-Ross: Bond Prices

- Conjecture that the log price of an $n+1$-period bond can be expressed as:

$$P_{n,t} = \exp(-n y_{n,t}) = \exp(A_n + B_n x_t)$$

- Note that $A_1 = 0$ and $B_1 = -1$ from the short rate equation

- Verify from the Euler equation

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- Verify from the Euler equation

$$\begin{aligned}
P_{n+1,t} &= E_t[\exp(m_{t+1} + p_{n,t+1})] \\
&= E_t[\exp(-y_{1,t} - \frac{1}{2}\lambda^2 x_t \sigma^2 - \lambda\sqrt{x_t}\varepsilon_{t+1} + A_n + B_n x_{t+1})] \\
&= E_t[\exp(-x_t - \frac{1}{2}\lambda^2 x_t \sigma^2 - \lambda\sqrt{x_t}\varepsilon_{t+1} + A_n)] \\
&\quad \times E_t[\exp(B_n(\delta + \rho\,(x_t - \delta) + \sqrt{x_t}\varepsilon_{t+1}))] \\
&= \exp\left(A_n + B_n\delta(1-\rho) + (B_n\rho - 1 - \lambda B_n \sigma^2 + \frac{1}{2}B_n^2\sigma^2)x_t\right)
\end{aligned}$$

Cox-Ingersoll-Ross: Bond Prices

- Solution for log price in CIR model for $A_0 = B_0 = 0$:

$$\begin{aligned} A_{n+1} &= A_n + B_n \delta(1-\rho) \\ B_{n+1} &= -1 + B_n \rho + \frac{B_n^2 \sigma^2}{2} - \lambda B_n \sigma^2 \end{aligned}$$

- Excess return on n-period bond:

$$\begin{aligned} r_{n+1,t+1} - y_{1,t} &= p_{n,t+1} - p_{n+1,t} - y_{1,t} \\ &= -\frac{B_n^2 \sigma^2}{2} x_t + \lambda B_n x_t \sigma^2 + B_n \sqrt{x_t} \varepsilon_{t+1} \end{aligned}$$

- Decompose excess return into:
  1. Jensen term
  2. Bond risk premium: positive if $\lambda < 0$, time-varying
  3. Unexpected bond return: heteroskedastic

Cox-Ingersoll-Ross: Properties

- One-factor model that can generate upward and downward sloping term structures
- Interest rates can no longer become negative (in continuous time limit)
- Yields and bond returns are heteroskedastic
- All bond yields are perfectly correlated, inherit autocorrelation of short rate
- Positive and time-varying bond risk premium, driven by same factor that drives yields

Affine Term Structure vs. C-CAPM

- Can write C-CAPM SDF in similar form:

$$M_{t+1} = \exp\left\{\log \beta - \gamma \log \Delta c_{t+1}\right\}$$

- If $\log \Delta c_{t+1} \overset{iid}{\sim} N(\mu, \sigma^2)$ then:

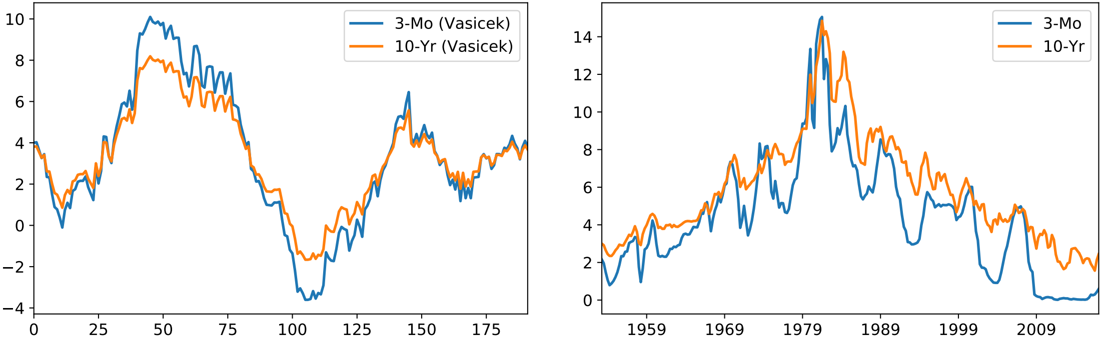
$$y_{1,t} = y_{n,t} = -\log \beta + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2$$

- If mean consumption growth is positively autocorrelated (e.g., $\mu_t = x_t$, long run risk): $\gamma > 0$ implies long bonds are hedges, negative risk premium.

- Positive risk premium requires negatively autocorrelated consumption growth or change to specification (e.g., stochastic inflation).

Vasicek Model vs. Data

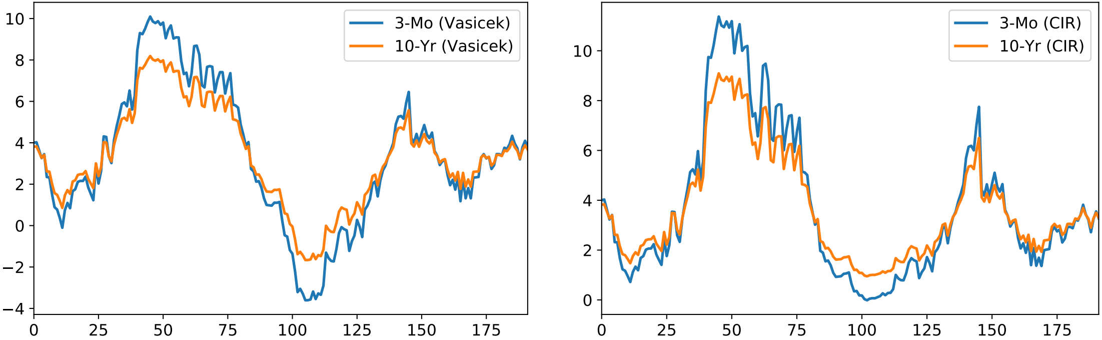
- Vasicek calibration: $\rho = 0.982$, $\sigma = 0.0057$, $\lambda = -1.4$ (all quarterly):



(a) Vasicek Model (b) Data

Vasicek Model vs. CIR Model

- CIR calibration: $\rho = 0.982$, $\sigma = 0.0304$, $\lambda = -1.4$ (all quarterly).
  - Same shocks applied to both models.



(a) Vasicek Model (b) CIR Model

Model 4: Affine Term Structure Model (ATSM)

- Standard $k-$factor essentially affine model in discrete time features (generalizes Vasicek):
  1. Multiple risk factors driving yields
  2. Time-varying risk premia
- Suppose we have k factors: $x_t \in R^k$:

$$x_{t+1} = \Gamma x_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}\left(0, \Sigma\right)$$

- These factors are latent, but we can recover them from yields

Model 4: ATSM

- SDF:

$$M_{t+1} = \exp\left(-y_{1,t} - \frac{1}{2}\Lambda_t' \Sigma \Lambda_t - \Lambda_t' \varepsilon_{t+1}\right)$$

- To keep the model affine:

$$\begin{aligned} y_{1,t} &= \delta_0 + \delta_1' x_t, \\ \Lambda_t &= \Lambda_0 + \Lambda_1 x_t \end{aligned}$$

- Identification: Dai and Singleton (2000)

ATSM: Bond Prices

- Solving the model is the same as before:

$$\begin{aligned} P_{1,t} &= \exp\left(-\delta_0 - \delta_1' x_t\right) \\ &= \exp\left(A_1 + B_1' x_t\right), \end{aligned}$$

  for $A_1 = -\delta_0,\ B_1' = -\delta_1'$.

- For any n :

$$\begin{aligned} P_{n,t} &= E_t\left(\exp\left(-y_{1,t} - \frac{1}{2}\Lambda_t'\Sigma\Lambda_t - \Lambda_t'\varepsilon_{t+1} + A_{n-1} + B_{n-1}' x_{t+1}\right)\right) \\ &= \exp\left(-y_{1,t} + \frac{1}{2}B_{n-1}'\Sigma B_{n-1} - \Lambda_t'\Sigma B_{n-1} + A_{n-1} + B_{n-1}'\Gamma x_t\right) \\ &= \exp\left(A_n + B_n' x_t\right), \end{aligned}$$

  where

$$\begin{aligned} A_n &= -\delta_0 + A_{n-1} + \frac{1}{2}B_{n-1}'\Sigma B_{n-1} - \Lambda_0'\Sigma B_{n-1}, \\ B_n' &= -\delta_1' + B_{n-1}'\left(\Gamma - \Sigma\Lambda_1\right) \end{aligned}$$

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ATSM: Bond Returns

- Bond returns

$$\begin{aligned} r_{n,t+1} &= p_{n-1,t+1} - p_{n,t} \\ &= A_{n-1} - A_n + B'_{n-1}\Gamma x_t - B'_n x_t + B'_{n-1}\varepsilon_{t+1} \\ &= \delta_0 - \frac{1}{2}B'_{n-1}\Sigma B_{n-1} + \Lambda'_0\Sigma B_{n-1} + \delta'_1 x_t + B'_{n-1}\Sigma\Lambda_1 x_t + B'_{n-1}\varepsilon_{t+1} \\ &= y_{1,t} - \frac{1}{2}B'_{n-1}\Sigma B_{n-1} + B'_{n-1}\Sigma\Lambda_t + B'_{n-1}\varepsilon_{t+1} \end{aligned}$$

- Components:
  1. Short rate
  2. Jensen term
  3. Bond risk premium (time-varying)
  4. Unexpected bond return

ATSM: Bond Risk Premium

- The bond risk premium can also be computed directly:

$$\log E_t \left[\exp\left(m_{t+1} + r_{n,t+1}\right)\right] = 0,$$

which implies:

$$E_t\left(r_{n,t+1}\right) - y_{1,t} + \frac{1}{2}\mathrm{Var}_t\left(r_{n,t+1}\right) + \mathrm{Cov}_t\left(m_{t+1}, r_{n,t+1}\right) = 0$$

The bond risk premium is given by:

$$-\mathrm{Cov}_t\left[m_{t+1}, r_{n,t+1}\right] = B'_{n-1}\Sigma\Lambda_t$$

Inverting Yields to Recover Factors

- We started with k latent factors
- However, we can "invert yields" to recover the factors
- Pick any k yields:

$$\begin{bmatrix} A_{n_1} \\ \dots \\ A_{n_k} \end{bmatrix} + \begin{bmatrix} B'_{n_1} \\ \dots \\ B'_{n_k} \end{bmatrix} x_t = -\frac{1}{n} \begin{bmatrix} y_{n_1,t} \\ \dots \\ y_{n_k,t} \end{bmatrix},$$

- Could invert to express all the factors as affine functions of the yields
  - This is important when you have observable macro factors!
  - In the data, macro variables are not linear combinations of bond yields (Joslin, Singleton, and Zhu, RFS 2011)

Properties

- Yield curve can take on any shape
- Risk premium is time-varying
- There is a k-factor structure in bond yields
- Interest rates can become negative
- Yields and bond returns are homoskedastic

Model 4: ATSM + Heteroskedasticity

- $k-$factor heteroskedastic Gaussian ATSM (generalizing CIR model):

$$x_{t+1} = \Gamma x_t + V(x_t)^{1/2}\varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0, I)$$

- $V(x)$ is a diagonal matrix with entries $V_{ii}(x_t) = \alpha_i + \beta_i x_t$
- The SDF is given by (constant market prices of risk)

$$M_{t+1} = \exp\left(-y_{1,t} - \frac{1}{2}\Lambda' V(x_t)\Lambda - \Lambda' V(x_t)^{1/2}\varepsilon_{t+1}\right)$$

- See Duffie-Kan (1996)

Pricing Real Bonds

- Most term structure models formulated in nominal terms
- TIPS data available since 1997, but initially illiquid
- Once we specify an inflation process, we can price real bonds:

$$\begin{aligned} P^R_{n,t} &= E_t\left[M_{t+1}P^R_{n-1,t+1}\frac{\Pi_{t+1}}{\Pi_t}\right] \\ &= E_t\left[M^R_{t+1}P^R_{n-1,t+1}\right], \end{aligned}$$

$$M_{t+1}\frac{\Pi_{t+1}}{\Pi_t} = M^R_{t+1}$$

- To get closed-form solutions, we can model inflation $\pi_{t+1} = \log(\Pi_{t+1}/\Pi_t)$:

$$\pi_{t+1} = \zeta_0 + \zeta_1' x_t + \sigma_\pi' \varepsilon_{t+1},$$

Empirics of the Term Structure: Motivation

- Affine models link risk factors to yields.
  - How many factors do we need to describe bond yields?
  - What about bond returns?
- The failure of the Expectations Hypothesis implies that bond risk premia vary over time.
- Can we link equity and fixed income models?

Number of Factors

- Yields are highly correlated
- For instance, correlation matrix for the 1y,...,5y yields (Fama-Bliss):

	1-Yr	2-Yr	3-Yr	4-Yr	5-Yr
1-Yr	100.0%	99.5%	98.7%	97.8%	97.1%
2-Yr	99.5%	100.0%	99.8%	99.3%	98.8%
3-Yr	98.7%	99.8%	100.0%	99.8%	99.5%
4-Yr	97.8%	99.3%	99.8%	100.0%	99.9%
5-Yr	97.1%	98.8%	99.5%	99.9%	100.0%

- How many factors are sufficient?
- Principal components analysis is the way to reduce the number of factors by minimizing the loss of information (in a statistical sense!)

Number of Factors: Principal Components

- First principal component solves:

$$\max_{q_1' q_1 = 1} var(q_1' y),$$

- Second principal component solves:

$$\max_{q_2' q_2 = 1, q_1' q_2 = 0} var(q_2' y),$$

- You can find the q's fast using an eigenvalue decomposition of $\Sigma = var(y)$:

$$\Sigma = Q \Lambda Q',$$

with Λ diagonal and $Q'Q = QQ' = I \implies Q^{-1} = Q'$

- Equivalent to SVD procedure.

Number of Factors: Principal Components

- Define factors as:

$$x_t = Q^{-1} y_t = Q' y_t$$

- Then we have:

$$var\left(x_t\right) = \Lambda$$

  hence, the factors are orthogonal and the variance is given by the diagonal elements of Λ
- The proportional variance of factor j is given by:

$$\Lambda_{jj} / \sum_i \Lambda_{ii}$$

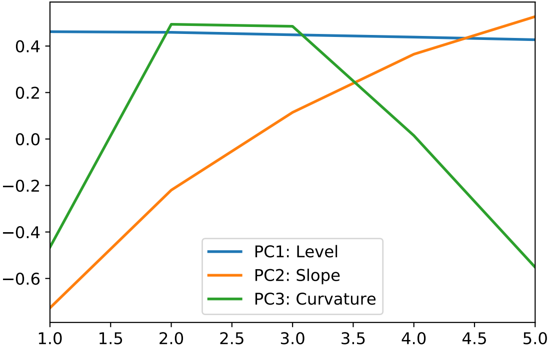
- For the same data, this ratio reads:

1-Yr	2-Yr	3-Yr	4-Yr	5-Yr
99.20%	0.75%	0.03%	0.01%	0.01%

- Three factors explain pretty much all variation in the cross-section of bond yields

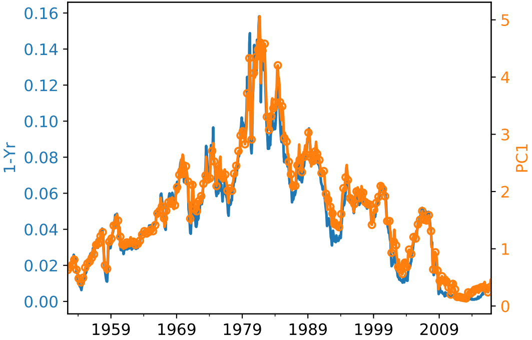
Number of Factors: Principal Components

- The rows of Q' illustrate what happens to all yields if one of the factors increases

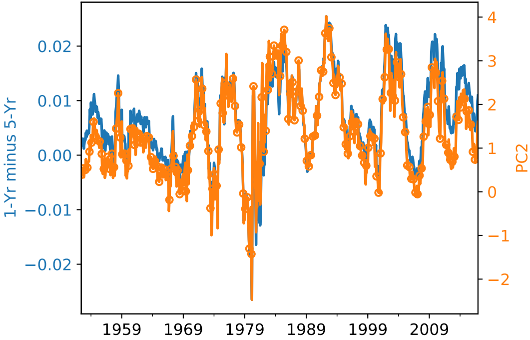


Number of Factors: Principal Components

- Level factor explains short rate.
- Slope factor explains spread.



(a) Short Rate vs. Level Factor



(b) Spread vs. Slope Factor

Number of Factors: Principal Components

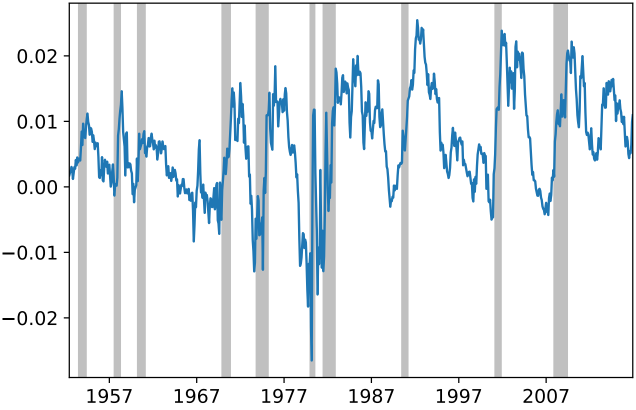
- Note that this is based on a statistical procedure using yields
- There could be factors that do not show up in yields, or only weakly, but that are important for bond returns and volatilities.
- More broadly, we want to understand why there is this factor structure and how it relates to technology, preferences, monetary policy, ...

Bond Risk Premia

- There is a large literature showing that Expectations Hypothesis fails (e.g., Fama-Bliss, 1987; Campbell-Shiller, 1991)
    - Yield spread does not sufficiently predict higher future short rate changes
    - Yield spread predicts long rate changes with the wrong sign
    - Forward rates do not sufficiently predict future short rates
- All these failures suggest that bond risk premia vary over time.
- Bond market variables may proxy for risk premia $\implies$ predictability.
    1. Yield spread: $y_{long,t} - y_{short,t}$.
    2. Forward spread: $f_t^{n \to n+1} - y_{1,t}$.

Yield Spreads

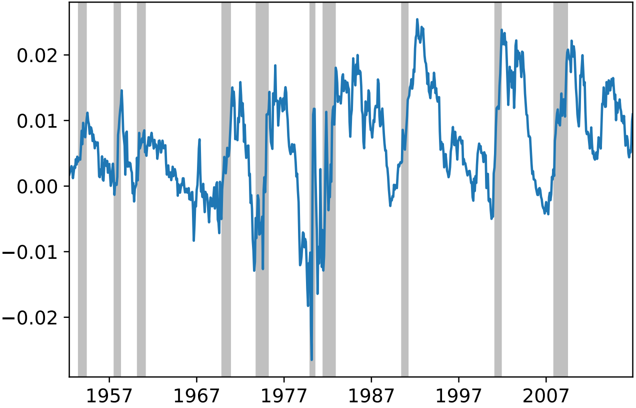
- Fama-Bliss (1987) and Campbell-Shiller (1991) show that yield spreads forecast future excess bond returns ($R^2 \simeq 15\%$)



5-Yr minus 1-Yr yield spread. Monthly Fama-Bliss data from 1952:6 - 2016:12.

Yield Spreads

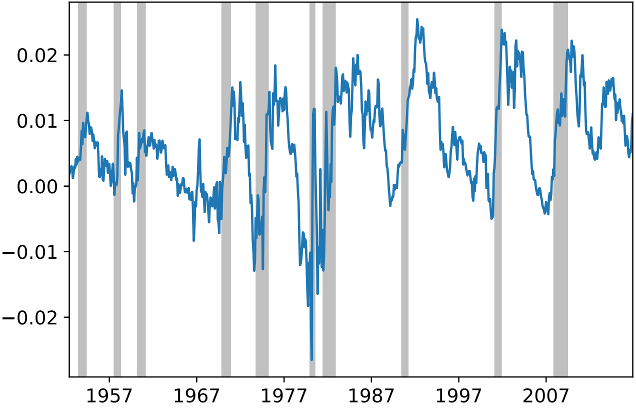
- Yield spreads are very low, often negative, prior to a recession



5-Yr minus 1-Yr yield spread. Monthly Fama-Bliss data from 1952:6 - 2016:12.

Yield Spreads

- Suggests that bond risk premia are counter-cyclical: low before a recession, and rising during a recession



5-Yr minus 1-Yr yield spread. Monthly Fama-Bliss data from 1952:6 - 2016:12.

Forward Rates as Predictor

- If the EH holds, then expected excess returns should be zero for all bonds, cannot be predicted by the forward spread $\left(f_t^{n\to n+1} - y_{1,t}\right)$.
- Lets run the following regression and test $H_0 : \gamma_{n+1,1} = 0, \ \forall n$

$$r_{n+1,t+1} - y_{1,t} = \gamma_{n+1,0} + \gamma_{n+1,1}\left(f_t^{n\to n+1} - y_{1,t}\right) + \varepsilon_{t+1}, \ n = 1,2,3,4$$

n (years)	2	3	4	5
$\gamma_{n+1,1}$	0.757	0.978	1.226	0.927
SE (NW, 12)	(0.223)	(0.302)	(0.368)	(0.435)
R^2	9.32%	9.62%	11.38%	50.23%

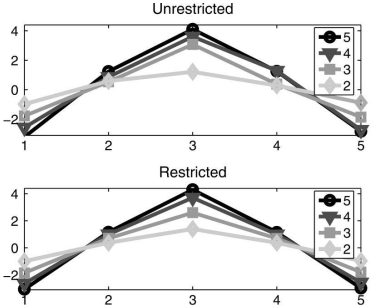
Monthly Fama-Bliss zero-coupon data from CRSP for 1952.6-2008.12.

Predicting Bond Excess Returns Using Forward Rates

- Cochrane and Piazzesi (AER, 2005): why not use all forward rates to predict excess returns?
- First, they regress excess returns of different horizons $n = 1, \cdots, 5$ on all lagged forward rates:
$$r_{n,t+1} - y_{1,t} = \alpha_n + \beta_n^1 y_{1,t} + \beta_n^2 f_t^{1\to 2} + \beta_n^3 f_t^{2\to 3} + \beta_n^4 f_t^{3\to 4} + \beta_n^5 f_t^{4\to 5} + \varepsilon_{n,t+1}$$
- Then, they extract a measure of bond risk premia from forward rates

Same function of forward rates predict HPR

- Let's plot the β_n's (top panel)



- All excess returns seem to have the same pattern, just stretched out
- Suggests one common factor in bond risk premia

Cochrane-Piazzesi: Two-Step Estimation

- First step: Extract common factor in bond risk premia as:

$$\frac{1}{4}\sum_{n=2}^{5} r_{n,t+1} - y_{1,t} = \alpha + \gamma^1 y_{1,t} + \gamma^2 f_t^{1\to 2} + \gamma^3 f_t^{2\to 3} + \gamma^4 f_t^{3\to 4} + \gamma^5 f_t^{4\to 5} + \varepsilon_{t+1},$$

- Define the Cochrane-Piazzesi factor as:

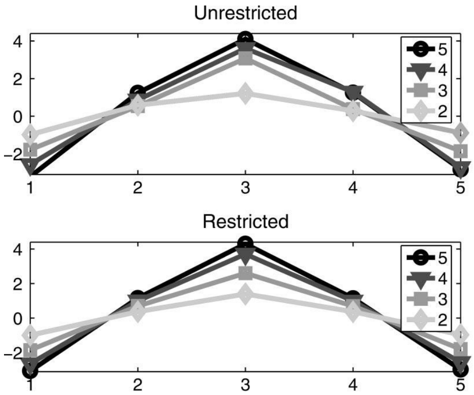
$$CP_t = \hat{\alpha} + \hat{\gamma}' f_t$$

- Second step:

$$r_{n,t+1} - y_{1,t} = b_n \underbrace{\left(\hat{\alpha} + \hat{\gamma}' f_t\right)}_{CP_t} + \varepsilon_{n,t+1}.$$

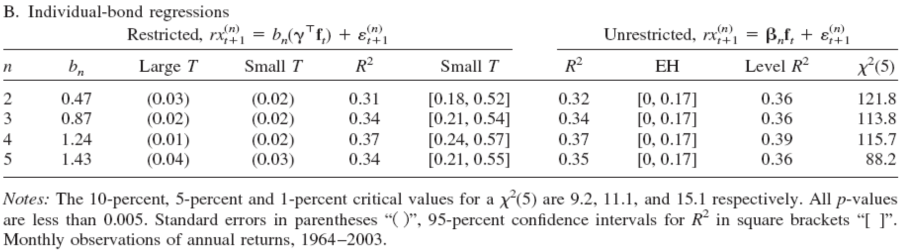
One-Factor Estimates

- Focusing on one common factor hardly reduces predictability (bottom panel)



Predictability R^2s

- This hardly reduces predictability



B. Individual-bond regressions

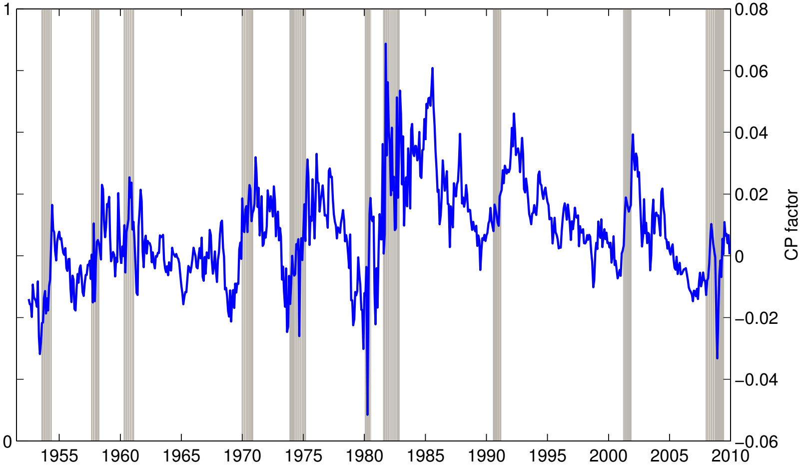
	Restricted, $rx_{t+1}^{(n)} = b_n(\gamma^\top \mathbf{f}_t) + \varepsilon_{t+1}^{(n)}$					Unrestricted, $rx_{t+1}^{(n)} = \beta_n \mathbf{f}_t + \varepsilon_{t+1}^{(n)}$			
n	b_n	Large T	Small T	R^2	Small T	R^2	EH	Level R^2	$\chi^2(5)$
2	0.47	(0.03)	(0.02)	0.31	[0.18, 0.52]	0.32	[0, 0.17]	0.36	121.8
3	0.87	(0.02)	(0.02)	0.34	[0.21, 0.54]	0.34	[0, 0.17]	0.36	113.8
4	1.24	(0.01)	(0.02)	0.37	[0.24, 0.57]	0.37	[0, 0.17]	0.39	115.7
5	1.43	(0.04)	(0.03)	0.34	[0.21, 0.55]	0.35	[0, 0.17]	0.36	88.2

Notes: The 10-percent, 5-percent and 1-percent critical values for a $\chi^2(5)$ are 9.2, 11.1, and 15.1 respectively. All p-values are less than 0.005. Standard errors in parentheses "( )", 95-percent confidence intervals for R^2 in square brackets "[ ]". Monthly observations of annual returns, 1964–2003.

- Bottom-line: One factor drives most of the variation in bond risk premia (and its not the Level factor like in CIR)

$$\begin{aligned}\Lambda_t &= \Lambda_0 + \Lambda_1 x_t \\ &\simeq \tilde{\Lambda}_0 + \tilde{\Lambda}_1 CP_t\end{aligned}$$

CP Factor



CP computed from Fama-Bliss data. Monthly CRSP data for 1952.6-2009.12.

Yields vs Returns

- There is information in the fourth and fifth principal components of yields that, while unimportant to explain variation in the cross-section of yields, is very important for understanding the variation in bond returns.
- In fact, if we orthogonalize CP to the first three PCs of yields, we retain most predictive ability.
- Duffee (2011) refers to this as *hidden factors* in the term structure of yields
- Ludvigson and Ng (2008) show that macro-economic factors have predictive ability for future excess bond returns over and above that for yields
  - Use principal components to extract common factors from large panel of macro time series
  - Yields do not fully capture that macro information, but it is relevant for bond excess returns

Linking Equity and Fixed Income Models

- This factor not only forecasts bond returns, it also has some forecasting power for stocks

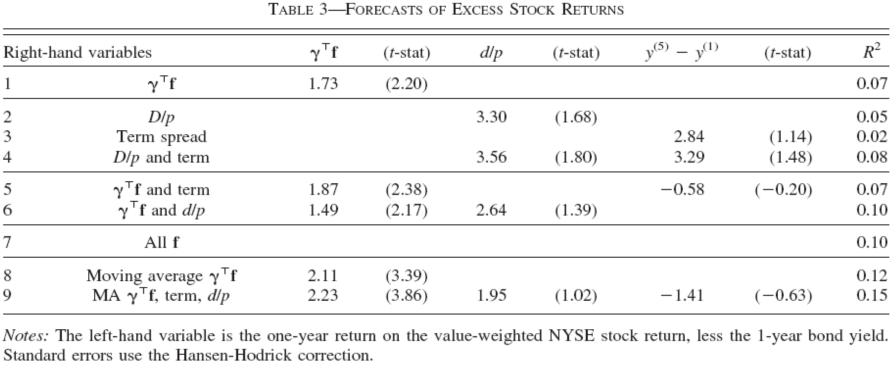


TABLE 3—FORECASTS OF EXCESS STOCK RETURNS

Right-hand variables		$\gamma^\top \mathbf{f}$	(*t*-stat)	*d*/*p*	(*t*-stat)	$y^{(5)} - y^{(1)}$	(*t*-stat)	R^2
1	$\gamma^\top \mathbf{f}$	1.73	(2.20)					0.07
2	*D*/*p*			3.30	(1.68)			0.05
3	Term spread					2.84	(1.14)	0.02
4	*D*/*p* and term			3.56	(1.80)	3.29	(1.48)	0.08
5	$\gamma^\top \mathbf{f}$ and term	1.87	(2.38)			−0.58	(−0.20)	0.07
6	$\gamma^\top \mathbf{f}$ and *d*/*p*	1.49	(2.17)	2.64	(1.39)			0.10
7	All **f**							0.10
8	Moving average $\gamma^\top \mathbf{f}$	2.11	(3.39)					0.12
9	MA $\gamma^\top \mathbf{f}$, term, *d*/*p*	2.23	(3.86)	1.95	(1.02)	−1.41	(−0.63)	0.15

Notes: The left-hand variable is the one-year return on the value-weighted NYSE stock return, less the 1-year bond yield. Standard errors use the Hansen-Hodrick correction.

Term Structure of Equity

- Can assets other than bonds have a term premium? Yes!
  - For a given risky cash flow process, compare price of receiving realization at different horizons.
- Equivalent of zero-coupon bond is zero-coupon equity: asset that pays dividend n periods from now.
  - Value stocks pay more dividends in the short run: more weight on short-duration equity.
  - Growth stocks pay more dividends in the long run: more weight on long-duration equity.
- Value premium $\implies$ term structure of equity is downward sloping.

Lettau and Wachter (2007)

- Fundamental shocks $\varepsilon_{t+1} \sim N(0, I_k)$.
- Log aggregate dividend growth process:

$$\begin{aligned} \Delta d_{t+1} &= z_t + \sigma_d' \varepsilon_{t+1} \\ z_{t+1} &= (1 - \phi_z) g + \phi_z z_t + \sigma_z' \varepsilon_{t+1}. \end{aligned}$$

- Stochastic discount factor (only dividend risk priced):

$$M_{t+1} = \exp\left\{ -r^f - \frac{1}{2} x_t^2 - x_t \varepsilon_{d,t+1} \right\}$$

for risk tolerance x_t and dividend risk $\varepsilon_{d,t+1}$ defined by

$$\begin{aligned} x_{t+1} &= (1 - \phi_x) \bar{x} + \phi_x x_t + \sigma_x' \varepsilon_{t+1} \\ \varepsilon_{d,t+1} &= \frac{\sigma_d' \varepsilon_{t+1}}{||\sigma_d||} \end{aligned}$$

Pricing Zero-Coupon Equity

- Conjecture the functional form

$$\frac{P_{n,t}}{D_t} = \exp\left\{ A_n + B_{x,n} x_t + B_{z,n} z_t \right\}.$$

- Solution:

Pricing Zero-Coupon Equity

- Conjecture the functional form

$$\frac{P_{n,t}}{D_t} = \exp\left\{A_n + B_{x,n}x_t + B_{z,n}z_t\right\}.$$

- Solution:

$$\begin{aligned}
B_{z,n} &= \frac{1-\phi_z^n}{1-\phi_z} \\
B_{x,n} &= B_{x,n-1}\left(\phi_x - \sigma_x' \frac{\sigma_d}{||\sigma_d||}\right) - (\sigma_d' + B_{z,n-1}\sigma_z')\frac{\sigma_d}{||\sigma_d||} \\
A_n &= -r^f + g + \frac{1}{2}\sigma_{n,t}'\sigma_{n,t} + A_{n-1} + B_{x,n-1}(1-\phi_x)\bar{x} \\
\sigma_{n,t} &= \sigma_d + B_{z,n-1}\sigma_z + B_{x,n-1}\sigma_x.
\end{aligned}$$

Increasing in expected dividend growth z_t, decreasing in risk price x_t for $\sigma_x'\sigma_d$ and $\sigma_z'\sigma_d \simeq 0$.

Market Returns

- Holding period return to zero-coupon equity:

$$r_{n,t+1} - r^f - \frac{1}{2}\sigma_{n,t}'\sigma_{n,t} = \sigma_{n,t}' \frac{\sigma_d'}{||\sigma_d||} x_t$$

- Aggregate market *PD* ratio and market returns:

$$\frac{P_t^m}{D_t} = \sum_{n=1}^{\infty} \frac{P_{n,t}}{D_t}$$

$$R_{t+1}^m = \frac{(P_{t+1}^m / D_{t+1}) + 1}{P_t^m / D_t} \frac{D_{t+1}}{D_t}$$

- Price of firm that pays deterministic shares $s_{j,t}$ of aggregate dividend:

Firm Returns

- Price of firm that pays deterministic shares $s_{j,t}$ of aggregate dividend:

$$P^F_{j,t} = \sum_{n=1}^{\infty} s_{j,t+n} P_{n,t}$$

- Assume N firms who rotate through the same deterministic sequence $(\bar{s}_1, \ldots, \bar{s}_N)$ so that

$$s_{j,t} = \bar{s}_{(t \bmod N)+j}$$

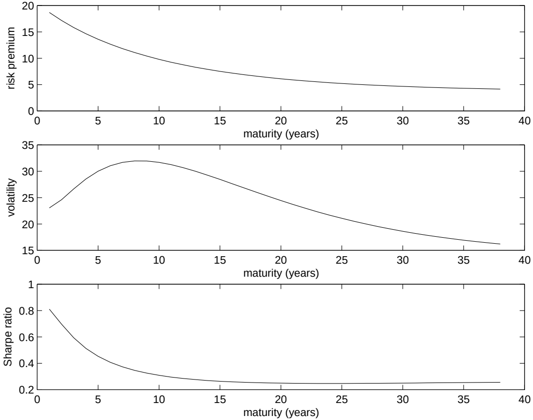
- Define shares so that

$$\bar{s}_{i+1} = \gamma \left(1 - \frac{i-1}{N/2-1} \right) \bar{s}_i$$

Results

- Reproduces asset pricing moments, value premium, downward sloping term structure of equity.

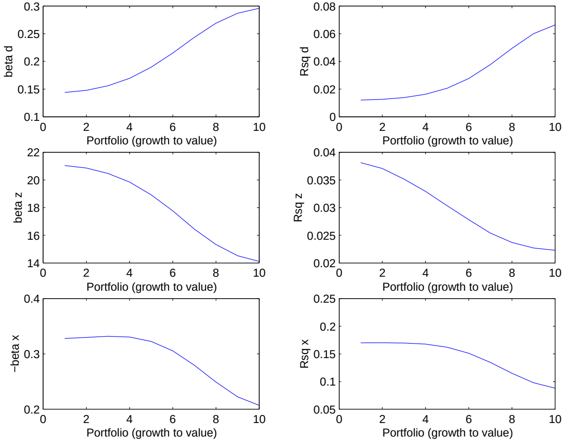
Figure 3: Properties of Zero-Coupon Equity



Results

- Value stocks more exposed to ε_d, less exposed to ε_z, ε_x.

Figure 8: Regressions on Fundamental Shocks



Results

- How are these risks priced?
- Dividend risk ε_d has positive price:
  - Investors have high MU when dividends low.
  - Value stocks more exposed to this risk.
- Dividend growth risk ε_z has negative price:
  - Calibration: ε_z negatively correlated with ε_d.
  - Stocks that load on z_t (e.g., growth stocks) are hedges.
- Sentiment risk ε_x is not priced.
  - Changes in risk tolerance uncorrelated with fundamentals.
- Result: value stocks have higher risk premium!

Lettau and Wachter (2011)

- Why is the term structure of equity downward sloping while the term structure of bonds is upward sloping?
- Extend model to add inflation process

$$\begin{aligned}
\Delta\pi_{t+1} &= q_t + \sigma_\pi' \varepsilon_{t+1} \\
q_{t+1} &= (1-\phi_q)\bar{q} + \phi_q q_t + \sigma_q' \varepsilon_{t+1}
\end{aligned}$$

and risk-free rate process

$$\begin{aligned}
r_{t+1}^f &= (1-\phi_r)\bar{r}^f + \phi_r r_t^f + \sigma_r' \varepsilon_t \\
M_{t+1} &= \exp\left\{ -r_{t+1}^f - \frac{1}{2}x_t^2 - x_t \varepsilon_{d,t+1} \right\}
\end{aligned}$$

Stock vs. Bond Term Structure

- Key is calibration of shock correlations.
- Negative correlation between ε_d and ε_r: yields low when dividends high $\implies$ premium for long-term real bonds.
- Negative correlation between ε_d and persistent inflation: nominal bonds risky, especially long-term ones.

Table 2: Conditional cross-correlations of shocks

Variable	$\Delta\pi_t$	z_t	q_t	r^f_{t+1}	x_t
Δd_t	-0.30	-0.83	-0.30	-0.30	0
$\Delta\pi_t$		0	1.00	0	0
z_t			0	0	0.35
q_t				0	0
r^f_{t+1}					0

Further Reading

- Role of Monetary Policy
  - Ang, Dong, and Piazzesi (2007): ”No-Arbitrage Taylor Rules”
  - Piazzesi (2005): ”Bond Yields and the Federal Reserve”
  - Palomino (2008): ”Bond Risk Premia and Optimal Monetary Policy”
  - Gallmeyer, Hollifield, Palomino, and Zin (2009, WP): ”Term Premium Dynamics and the Taylor Rule”
- Consumption-Based Models
  - Wachter (2005, JFE): ”A Consumption-Based Model of the Term Structure of Interest Rates”
  - Bansal and Shaliastovich (2013, RFS): ”A Long-Run Risks Explanation of Predictability Puzzles in Bond and Currency Markets”
  - Gabaix (2012, QJE): ”Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance”